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Developing a Representation of Simple Voting Games Within Category Theory

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Philosophy PhD - Developing a Representation of Simple Voting Games Within Category Theory

Simon David Terrington

June 29, 2012

Abstract

Simple voting games (SVGs) are mathematical idealisations of decision-making by a council or board for example the EU Council of Ministers or the UN Security Council.

The theory of SVGs includes structure-preserving mappings. Until now, these have not been organised in a category.

We start in the most natural way, with the objects of the category being SVGs conceived as sets of sets of voters and arrows being isomorphisms, bloc formation and inclusion maps.

An alternative category, or sequence of categories, of SVGs, is simple to define. C_0 is the category with two objects and a single non-identity arrow between them. C_{n+1} is the arrows category of C_n. Encouragingly, the category-theoretic duals, products and coproducts correspond to duals, meets (products) and joins (coproducts) in SVGs. This category is a partial order. We develop a new notation for SVGs and have almost immediate proof of substantial results, for example the construction of the constant-sum extension of a game and the fact that the Banzaf-Penrose measure for a bloc of two voters is equal to the sum of the Banzaf-Penrose measures for the two voters.

The C_n are also lattices. We can find the bipartitions as a sublattice which is also Boolean algebra.

Lattice homomorphisms between the C_n correspond to structure-preserving mappings of SVGs. We can build another category with these.

We also have a bijective mapping between SVGs and ordered pairs of a simplicial complex and its Alexander dual. This connects the theory of SVGs with topology.

Contents

1 Introduction and Basic Definitions

Simple voting games model decision making as seen in bodies such as the EU Council of Ministers, UN, US Electoral College and public limited companies. In all of these situations, the various members of the group (ministers, delegates or shareholders) are asked to vote for a motion. If enough of them vote 'yes' then the motion is passed.

A simple voting game is currently conceived as a set of sets. It consists of all of those sets of voters that pass the motion. We assume that adding voters to a set can only leave the result the same or make it better; it is not possible to turn a group of voters (or coalition) from winning to losing by adding voters. This condition is referred to as monotonicity.

In this paper, we will show that there is another way to conceive of SVGs other than as a set of sets. This is helpful in terms of deriving theorems.

To begin with, we follow Fesenthal and Machover [1, Definition 2.1.1] and exclude the game for which every coalition wins and the one for which every coalition loses. In practical decision making these games do not occur. Later, we will be forced to introduce them because it makes our mathematical structures work. Perhaps we could compare their role to the square root of minus one or the one-point compactification of the plane. They don't occur naturally in practice but they are required to 'complete' the mathematical system.

In many branches of mathematics, structure-preserving functions play an important role. Examples of these include homomorphisms in group theory, continuous maps in topology and linear functions in the theory of vector spaces. Structure-preserving functions allow us to understand substructures but also build objects such as products (in many branches of mathematics) and identification spaces in topology. One could even say that the structure-preserving mappings define a branch of mathematics. The characteristic feature of geometry is that two objects are considered to be the same or congruent if one can be turned into another by isometries (a combination of translations, reflections and rotations). Two objects are considered equivalent topologically if one can be mapped to another through bending and stretching without tearing (that, of course, it how you recognise a topologist at a tea party - she can't tell the different between a tea cup and a doughnut). Sets are considered equal if they have the same members but equipollent (isomorphic within the category of sets) if they have the same cardinality.

Category theory is about doing mathematics with these structure-preserving functions. Theorems in category theory have corresponding theorems in the category of each branch of mathematics.

We can find relationships between categories (in terms of functors) that bring new insights and allow us to transfer findings between different branches of mathematics. The most obvious example being the fusion of group theory and topology in algebraic topology.

In the case of simple voting games we have structure-preserving mappings: isomorphisms, bloc formation and inclusion mappings. But these have not been organised into a category. So there is an opportunity to do this, understand the properties of the resulting category and look for relationships with other categories.

The question then becomes 'What are the objects and arrows of this category?'. We assume that the objects are SVGs but conceived how?

We start in the most natural way, with the objects being SVGs conceived as sets of sets of voters and arrows being isomorphisms, bloc formation and inclusion maps. We can build and understand this category but it does not seem to significantly deepen our understanding of simple voting games. Also products and coproducts in this category don't correspond to the way products and coproducts are defined in the theory of SVGS. The category also lacks a way of expressing the game-theoretic dual.

The next category, or sequence of categories, have very simple definitions. C_0 is one of the simplest categories - the one with two objects and a single non-identity arrow between them. C_{n+1} is the arrows category of C_n. We can interpret C_n as consisting of all games with n voters. Encouragingly, duals, products and coproducts correspond to what we already have in simple voting games but we do not have structure-preserving mappings between objects. Instead, an arrow 'says' that the codomain game is bigger than the domain, that is, every division which wins the domain also wins the codomain. There is no more than one arrow between any two objects so this category is actually a partial order. Despite theses limitations, this approach allows us to develop a novel notation for displaying simple voting games and calculate important measures of voting power including the Penrose-Banzaf index. With this conceptual toolkit we can also do quite a lot of mathematics, finding a simple algorithm to list the SVGs (Comment 434) and presenting almost immediate proof of substantial results such as the construction of the constant-sum extension of a game (Comment 451) and the fact that the Banzaf-Penrose measure for a bloc of two voters is equal to the sum of the Banzaf-Penrose measures for the two voters (Comment 452). Of course, it is not just that the proofs are quicker; they also give a new understanding of the results.

Not only are these categories partially ordered sets, they are also lattices. On top of this, we can find two Boolean algebras as sublattices of each C_n. These Boolean algebras (both) correspond to the Boolean algebra of finite sets and can be conceptualised as bipartitions in a natural way. A bipartition winning a game is represented very simply, by an arrow from the bipartition to the game. We can also find objects that correspond to the voters. The idea of them voting 'yes' in a bipartition is represented by an arrow from the bipartition to them.

Lattice homomorphisms between these categories of SVGs turn out to correspond to structure-preserving mappings of SVGs so we can build a category which has the C_n as objects and lattice homomorphisms (structure-preserving mappings) as the arrows.

Finally, I have found a bijective mapping between simple voting games and ordered pairs of a simplicial complex and its Alexander dual. Potentially this connects the theory of SVGs with the large and dynamic discipline of topology.

This connection with topology has not been recognised in the literature, never mind exploited. Why is this? I would suggest that it is a consequence of the game-theoretic origin of the subject. There is a focus on winning coalitions rather than losing coalitions which is, of course, where we need to look for this result.

This result came late in the process. I had time to prove one theorem: if a games is weighted (or linear) and the edge group of either of the corresponding simplicial complexes are non-trivial then they contain an empty triangle.

There is one thing to say about exposition. I have doggedly proved every result and referenced every definition. This can be annoying not least for my respected supervisor. There are two reasons for doing this. The first is that I am inexperienced in this field and so it is hard to get people (including myself) to trust my assertions. Proof sets everybody's mind at rest. The second is that, I think that whether or not a proof is 'trivial' is one of the most subjective questions. When I was an undergraduate, people would describe results as obvious when I had no idea what was going on. On the other hand, there are things I can see and it is very hard to explain to others without rigorous proof. The problem, for the reader, is that there is pressure to check many proofs of results that are obvious and this becomes tiresome. I'm sorry about this. If the result is obvious, I would encourage the reader to skip the proof. Things have got better; the early drafts also included the proofs of dual theorems. These have been removed.

Definition 1. Given a set ($V \neq \emptyset$) of voters, a *simple voting game (SVG)* G_V is an ordered pair (V, G) where G is a subset of the power set of V such that:

1. If $A \in G$ and $A \subseteq B$ then $B \in G$.

2. G is not empty (Or, equivalently, $V \in G$).

3. G is not equal to the whole power set of V (Or, equivalently, $\emptyset \notin G$).

This is the definition that is given by Felsenthal and Machover ([1, Definition 2.1.1]).

Comments 2. The fact that we have specified $V \in G$ and $\emptyset \notin G$ means that V cannot be equal to $\emptyset$. Later, we will relax this condition.

Definition 3. Given games G_V and H_W, we say that H_W *is isomorphic to* G_V iff there is a one-to-one, onto function $i : V \to W$ such that for every $A \subseteq V$, $i(A) \in H \iff A \in G$.

Definition 4. Given games G_V and H_W, we say that H_W *is formed from* G_V *under bloc formation* iff there is an onto function $b : V \to W$ such that for every $A \subseteq W$, $b^{-1}(A) \in G \iff A \in H$

Definition 5. Given games G_V and H_W, we say that G_V *is a subgame of* H_W iff

$V \subseteq W$ and $G = \{A : (A \in H) \wedge (A \subseteq V)\}$ for V a winning coalition of H or, equivalently, iff there is an inclusion mapping $i : V \to W$ such that $i(V) \in H$ and for every $A \subseteq W$, $i^{-1}(A) \in G \iff A \in H$

Comments 6. The restriction that V is winning in H_W or, equivalently, that $i(V) \in H$ stops us obtaining the game that always loses as a subgame. This game is explicitly ruled out by Definition 1. This is a restriction that we will want to relax later.

Definition 7. Given games G_V and H_W, we say that G_V *is a reduced game of* H_W iff

$V \subseteq W$ and $G = \{A : (A \subseteq V) \wedge (A \cup (W - V) \in H)\}$ for $W - V$ not a winning coalition of H or, equivalently, iff there is an inclusion mapping $i : V \to W$ such that $(W - i(V)) \notin H$ and for every $A \subseteq W$, $A \in G \iff i(A) \cup (W - i(V)) \in G$

Comments 8. The restriction that $W - V$ is not winning in H_W or, equivalently, that $(W - i(V)) \notin H$ stops us obtaining the game that always wins as a reduced game. This game is explicitly ruled out by Definition 1. This is a restriction that we will want to relax later.

Comments 9. If $V \subseteq W$ and H_W is a game, then the subgame is the game that remains if all of the members of $W - V$ vote 'no'. The reduced game is what results if all the members of $W - V$ vote 'yes'.

Definition 10. I define a game G_V as *proper* iff for all $A \subseteq V$, $A \notin G$ or $(V - A) \notin G$

[3, Definition 1.3.3]

Definition 11. I define a game G_V as *strong* iff for all $A \subseteq V$, $A \in G$ or $(V - A) \in G$

[3, Definition 1.3.3]

Definition 12. If $A \in G$ and for all $B \subset A, B \notin G$ then we say that A is a *minimal winning coalition of* G_V

Comments 13. We can define a game by listing its minimum winning coalitions. This is a consequence of monotonicity.

Definition 14. Let G_V be an SVG. $v \in V$ is *a passer* iff $\{v\}$ is a winning coalition of G_V. [1, Definition 2.3.4]

Definition 15. Let G_V be an SVG. $v \in V$ is *a vetoer* iff v is a member of every winning coalition of G_V.

Definition 16. Let G_V be an SVG. $v \in V$ is *a dictator* iff $\{v\}$ is the only minimal winning coalition of G_V. [1, Definition 2.3.4].

Theorem 17. *Given an SVG, G_V, a voter $v \in V$ is a dictator iff it is a passer and a vetoer.*

Proof. Assume that v is a dictator. $\{v\}$ is a minimal winning coalition (Definition 16) and so $\{v\}$ is a winning coalition and v is a passer (Definition 14). Also if there is a winning coalition that does not include v then it must have a subset that is a minimal winning coalition and does not include v. This contradicts the fact that v is a dictator and $\{v\}$ is the only minimal winning coalition (Definition 16) and so every winning coalition includes v and v is a vetoer (Definition 15).

Assume that v is a passer and a vetoer. $\{v\}$ is a winning coalition (Definition 14). It is obviously minimal winning (because the only smaller set is $\emptyset$). There cannot be another minimal winning coalition because, as a vetoer, v is in every winning coalition (Definition 15). And so $\{v\}$ is the only minimal winning coalition and v is a dictator (Definition 16).

□

Definition 18. Let G_V be an SVG. $v \in V$ is *a dummy* iff v is not a member of a single minimal winning coalition of G_V. [1, Definition 2.3.4]

Definition 19. Let G_V be an SVG. *The Dual of G_V is defined as $G^*_V = (V, G^*)$ where $G^* = \{A : (V - A) \notin G_V\}$*

[1, Definition 2.3.2]

Theorem 20. *G_V^* is an SVG.*

Proof. $V \in G \implies V - V = \emptyset \notin G^*$

$\emptyset \notin G \implies (V - \emptyset) = V \in G^*$

Let us assume $B \subseteq A$ and $B \in G^*$. I need to show that $A \in G^*$

We know from the definition of a dual game (Definition 19) that $(V - B) \notin G$.

$B \subseteq A \implies (V - A) \subseteq (V - B)$.

Since G_V is monotonic (Definition 1), it must be that $(V - A) \notin G$.

By the definition of a dual game (Definition 19), we know that $A \in G^*$

□

Definition 21. A *Weighting System* on a finite set V is an ordered pair (q, w) where q is a real number and w is a mapping that assigns a non-negative real number to every $v \in V$ such that $0 < q \leq \Sigma w_v$. We extend the function from V to the real numbers to a function from the power set of V to the real numbers. For $S \subseteq V$, $wS = \Sigma_{x \in S} w_x$. The weighting system is said to be *normalised* if $wV = 1$.

[1, Definition 2.3.14]

Definition 22. The SVG $\{X \subseteq V : wX \geq q\}$ is the weighted voting game of the weighting system (q, w).

[1, Definition 2.3.14]

Comments 23. Weighted games have a simple structure; they can be entirely described by a vector of $|V| + 1$ real numbers (and in fact, we can arrange for these to be integers ([3, Page 6, Paragraph 5]). Games that cannot be expressed by a weighting system are often much more complex structurally.

Definition 24. If $G_V = (V, G)$ and $H_W = (W, H)$ are simple voting games then we define $(G \times H)_{V \cup W} = (V \cup W, G \times H)$ *where* $G \times H$ *is equal to the subset of the power set of* $V \cup W$ *such that* $S \in G \times H \iff ((S \cap V) \in G) \wedge ((S \cap W) \in H)$

If $V \cap W = \emptyset$ then the meet is called the product and denoted $G_V \wedge H_W$.

[1, Definition 2.3.12]

Definition 25. If $G_V = (V, G)$ and $H_W = (W, H)$ are simple voting games then we define $(G + H)_{V \cup W} = (V \cup W, G + H)$ *where* $G + H$ *is equal to the subset of the power set of* $V \cup W$ *such that* $S \in G + H \iff ((S \cap V) \in G) \vee ((S \cap W) \in H)$*.*

If $V \cap W = \emptyset$ then the join is called the sum and denoted $G_V \vee H_W$.

[1, Definition 2.3.12]

Definition 26. Given $G_V = (V, G)$, let us assume, for the sake of simplicity, that $V = \{1, 2, \dots v\}$ (If this is not the case then we can always find an isomorphism from V to the set of natural numbers less than or equal to $|V|$). For each $i \in V$ if we also have $(H_i)_{W_i}$.

We write *The composite of* $(H_1)_{W_1}, (H_2)_{W_2}, \dots, (H_v)_{W_v}$ *under* G_V *as* $G_V[(H_1)_{W_1}, (H_2)_{W_2}, \dots, (H_v)_{W_v}] = (\cup W_i, K)$.

If A is a subset of $\cup W_i$

$A \in K \iff \{i : A \cap W_i \in H_i\} \in G$

[1, Definition 2.3.12]

Comments 27. Meet (product) and join (sum) are special cases of composite games, for which $G_{\{v_1, v_2\}} = v_1 \wedge v_2$ (Definition 24) and $v_1 \vee v_2$ (Definition 25) respectively.

We can consider all of the W_j to be equal to $(\bigcup W_i)_{i=1}^{v}$ in Definition 26. This is achieved by taking members of $(\bigcup W_i)_{i=1}^{v}$ that are not in W_j and inserting them into H_{W_j} as dummies.

2 What are the Objects and Arrows of the Category of SVGs?

If we are to present SVGs as a category, the first question we need to ask is: 'What are the objects and arrows of this category?'

The most natural way to start is with SVGs (conceived as sets of sets) as objects and structure-preserving mappings between them as morphisms.

What do we include as structure-preserving mappings? At minimum we should include isomorphisms. The category of every algebraic structure allows isomorphisms as arrows. But these can only be a subset of the morphisms otherwise our category will simply express equivalence classes of isomorphic structures. What other morphisms should we allow? The two obvious candidates to examine are a surjective arrow to simulate bloc formation and an injective arrow from a subgame to a game.

We see something similar to the injective mapping that picks out a subgame in the categories such as groups, rings and vector spaces which have mappings to pick out subgroup, subrings and subspaces.

Bloc formation does not really have an analogue in algebraic structures. It corresponds to the idea of two (or more) voters grouping together and agreeing to act as one. This excludes the possibility of them voting in different directions but seems to keep the basic structure of the game the same.

Continuous maps from a topological space to its identification space and linear projection maps in the category of vector spaces have a similar feel to them. In the world of ultrafilters, these sorts of maps are called the Rudin-Keisler Projections [3, Page 24]. Of course, every SVG can be thought of as a union of filters with passers corresponding to ultrafilters.

These bloc formation arrows can simply be thought of as surjections on the voters with the winning sets of the domain always being the preimage of winning sets in the codomain. What would a category look like with just these bloc formation functions as arrows? The axioms would be satisfied. The composition of two surjections is a surjection. The identity function is a surjection. All the axioms obviously hold. But it's not a very interesting category. For example, there is no terminal element. The most likely candidate would be the game with one voter (the set with one element is terminal in the category of sets; the group with one object is terminal in the category of groups and the topological space with one object is terminal in the category of topological spaces). The problem is that there is only one game with n voters that has an arrow to the game with one voter, namely the unanimity game. For all of the others, we would like an arrow that maps all the voters in the domain onto the single voter in the codomain game but this arrow fails the condition $b^{-1}(A)$ wins $G_V \iff A$ wins H_W.

So what's the solution? We are trying to build SVGs as a concrete category. The objects are sets with an extra structure. The categories of groups and topological spaces are also concrete categories. Groups are sets along with an operation. Topological spaces are a set: S along with a set of subsets of S (the open sets). There seems to be a parallel between games and topological spaces. Games are a set of voters V and a set of subsets of V (the winning coalitions). In the category of topological spaces, the arrows are continuous functions. We know that the inverse image of an open set must be an open set but open sets don't necessarily map to open sets. So could we try a similar move here? Perhaps the appropriate condition on arrows is:

Definition 28. Given $g : V \to W$. I will refer to g as *conservative* (in the sense that such maps do not create winning coalitions in the range that were not there in the domain) $\iff$ $(A \in H_W \implies g^{-1}(A) \in G_V)$.

Comments 29. I am being being sloppy with language here. I describe a map $g : V \to W$ as being conservative when actually that status depends on the game: G_V and H_W. g is actually a map from (V, G) to (W, H). The images of members of G are determined by the images of their elements as members of V so they are usually suppressed but whether the map is conservative depends completely on G and H.

We see something similar in topology where people talk about functions between sets being continuous but, of course, this depends on the topology. There is not too much damage done because the topology is usually obvious from the context. I will try to make sure that the same is true with games.

This does give us a category with a terminal object, products and pullbacks but it doesn't have an initial object or coproducts. It is also not just colimits that are missing. Equalizers require injections (Of course, if a category has a terminal object, products and equalizers then it has all limits [4, P50]).

The natural step is to add injections as maps. This gives us:

Definition 30. $\mathbf{A}$ is the category with SVGs as objects. Arrows between SVGs (G_V and H_W) are functions $f : V \to W$ that are conservative.

Comments 31. It is necessary to check the axioms.

Given arrows $f : G_V \to H_W$ and $g : H_W \to K_X$, we need an arrow $gf : G_V \to K_X$.

To allow this in all cases, I need to relax the condition, on injections $i : G_V \to H_W$, that $i(V)$ is winning in H_W.

For example, if $X = \{1, 2, 3, 4\}$ and K_X has one minimal winning coalition: $\{1, 2\}$, $W = \{1, 2\}$ and both 1 and 2 are passers in H_W and $j : H_W \to K_X$ is the inclusion mapping (of W into X). $V = \{1\}$ and $i : V \to W$ is the identity inclusion mapping. The image of $\{1\}$ under ji is $\{1\}$ and that is not winning.

$$G_V \xrightarrow[i]{} H_W \xrightarrow[j]{} K_X$$

$$\{\{1\}\} \xrightarrow[i]{} \{\{1\}, \{2\}\} \xrightarrow[j]{} \{\{1, 2\}\}$$

When we relax the condition that $i(V)$ is winning, then for each set V, we then need to allow the game that always loses (the game with no winning coalitions) so that, given an injection, we can always carry out the operation of forming a subgame.

This raises the obvious question of whether, for each set V, we should include the game that always wins. I think we should for two reasons: first, it seems sensible that the category is closed under the operation of taking the dual and the dual of the game that always loses is the game that always wins. Second, if we don't have this game then we don't have an initial element for the category.

The only other thing that we need to do, to check that composition works, is to show that if Z is winning in K_X then $(gf)^{-1}(Z)$ is winning in G_V. Since g is an arrow then $g^{-1}(Z)$ is winning and since f is an arrow then $f^{-1}g^{-1}(Z)$ is winning. Of course, $(gf)^{-1} = f^{-1}g^{-1}$.

The other axioms are much less complicated. Identities $i : A \to A$ just come from the category of sets and clearly, the preimage of a winning set is winning. Associativity comes from the category of sets.

Theorem 32. **A** *has a terminal object.*

Proof. There is exactly one arrow from every set to the set with only one element (any one element set is terminal in the category of sets). We need the preimage of every winning coalition to be winning so the terminal game needs to have no wining coalitions, therefore, it must be the game that always loses. Otherwise, there would be no arrow from the game that always loses. □

Comments 33. As a matter of interest, we would still have had a terminal object if we did not include, injections and the games that always win and lose. In this case it would be the only (up to isomorphism) allowed game with one voter.

Theorem 34. **A** *has an initial object.*

Proof. The game with no voters that always wins is initial. Given any set V of voters, there is only one function from $\emptyset$ to V: the empty function.

Under this function, the preimage of every set is the empty set. Since this is winning then the empty function is an arrow.

□

Theorem 35. **A** *has all finite products.*

Proof. Let G_V and H_W be simple voting games.

Let $V \times W$ be the normal set-theoretic product of V and W.

Build a new game $K_{V \times W}$ by listing its minimal winning coaltions.

There are two types of minimal winning coalitions. First sets $\{(v, w) : (v \in M) \wedge (w \in W)\}$, for M a minimal winning coalition of G_V. Next, sets $\{(v, w) : (v \in V) \wedge (w \in N)\}$ for N a minimal winning coalition of H_W.

The projection maps are the set-theoretic projections $p_1 : V \times W \to V$ and $p_2 : V \times W \to W$. I need to show that these are conservative: that the preimage of a winning coalition is winning. Let us consider $p_1 : V \times W \to V$. A winning coalition of G_V must be a superset of a minimal winning coalition M. There is a corresponding minimal winning coalition of $K_{V \times W}$: $\{(v, w) : (v \in M) \wedge (w \in W)\}$ which is the preimage of M under p_1. The preimage of our original winning coalition must be a superset of the minimal winning coalition $\{(v, w) : (v \in M) \wedge (w \in W)\}$ and hence must also be winning. So p_1 is allowed as an arrow. The argument for p_2 is identical with $\{(v, w) : (v \in V) \wedge (w \in N)\}$ taking the place of $\{(v, w) : (v \in M) \wedge (w \in W)\}$.

Finally, let us start with an SVG L_Z and arrows m and n to G_V and H_W.

$$
\begin{tikzcd}
 & L_Z \arrow[dl, "m"'] \arrow[d, "g"] \arrow[dr, "n"] & \\
G_V & K_{V\times W} \arrow[l, "p_1"] \arrow[r, "p_2"'] & H_W
\end{tikzcd}
$$

We can now build an arrow g from L_Z to $K_{V\times W}$. It is simply the unique arrow that comes from the fact that $V \times W$ is the set-theoretic product of V and W. So $g(z) = (m(z), n(z))$. I need to check that the inverse images of winning coalitions of $K_{V\times W}$ are winning coalitions of L_Z.

Let $C \in K$. I need to show that $g^{-1}(C) = \{z \in Z : g(z) \in C\} \in L$.

To do this, I need to show that $g^{-1}(C) = m^{-1}p_1(C)$.

$z \in g^{-1}(C)$

$\iff g(z) \in C$

$\iff p_1(g(z)) \in p_1(C)$

$\iff m(z) \in p_1(C)$

$\iff z \in m^{-1}p_1(C)$

C is winning iff $p_1(C)$ is winning by the definition of $K_{V\times W}$. And so $m^{-1}(p_1(C))$ is winning because m is an arrow. Hence $g^{-1}(C) = m^{-1}p_1(C)$ is winning.

□

Comments 36. Let's look at some examples.

$\{\{1\},\{2\}\} \times \{\{3\},\{4\}\} =$

$\{\{(1,3),(2,3)\},\{(1,4),(2,4)\},\{(1,3),(1,4)\},\{(2,3),(2,4)\}\}$

$$\{\{1\},\{2\}\} \times \{\{3,4\}\} = \{\{(1,3),(1,4)\},\{(2,3),(2,4)\}\}$$
$$\{\{1,2\}\} \times \{\{3,4\}\} = \{\{(1,3),(1,4),(2,3),(2,4)\}\}$$

Comments 37. In this category, we do have products but they don't correspond to the game-theoretic meet or product (They have different voter-sets). We also do not have an operation that corresponds to taking the dual.

Theorem 38. *In* **A** *we have all finite coproducts.*

Proof. Given games G_V and H_W, the natural move is two build a game on $V+W$. I'll call it K_{V+W}

We need arrows $q_1 : G_V \to K_{V+W}$ and $q_2 : H_W \to K_{V+W}$.

We start with the injections $q_1 : V \to V+W$ and $q_2 : W \to V+W$ which come naturally from $V+W$.

The next thing to do is to work out the members of K. The fact that q_1 and q_2 are conservative restricts our choice - every member of K needs to have a member of G or H as its preimage of q_1 and q_2 respectively. The natural move is to make C a winning coalition of K iff its intersection with V (inverse image under q_1) is winning in G_V or its intersection with W (inverse image under q_2) is winning in H_W.

I need to check this object and these mapping satisfy the commuting conditions. That is given a game L_Z and arrows q_1 and q_2 such that this diagram commutes:

$$\begin{tikzcd}
 & L_Z & \\
G_V \arrow[ur, "m"] \arrow[r, "q_1"'] & K_{V+W} & H_W \arrow[ul, "n"'] \arrow[l, "q_2"]
\end{tikzcd}$$

We need a unique arrow u that makes this commute.

$$\begin{tikzcd}
 & L_Z & \\
G_V \arrow[ur, "m"] \arrow[r, "q_1"'] & K_{V+W} \arrow[u, "u"] & H_W \arrow[ul, "n"'] \arrow[l, "q_2"]
\end{tikzcd}$$

$u : V + W \to Z$ is given by the fact that $V + W$ is a direct sum. I just need to show that the associated map from K to L is conservative. We know that $m = uq_1$ and $n = uq_2$ Let X be a winning coalition of L_Z so $X \in L$.

I need to show that $Y = \{y : u(y) \in X\}$ is winning.

m is an arrow and so it is conservative. This means that $\{y : m(y) \in X\}$ is winning. We know that $q_1(\{y : m(y) \in X\})$ is winning because that is how we defined K.

Finally $q_1(\{y : m(y) \in X\}) = \{y : u(y) \in X\} = Y$ so Y is winning and u is conservative.

$\square$

Theorem 39. **A** *also has all equalisers.*

Proof. Let us start with a parallel pair of arrows.

$$\begin{tikzcd} G_V \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & H_W \end{tikzcd}$$

From the category of sets, we have an equaliser

$$\begin{tikzcd} E \arrow[r, "e"] & V \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & W \end{tikzcd}$$

I then define K to be as small as it could be to make e conservative. This also means that K is the subgame defined by e, which is monic ([4, Theorem 2.6])

So $X \in K \iff e(X) \in G$.

I need to show that $e : K_E \to G_V$ is an equaliser.

$$\begin{tikzcd} K_E \arrow[r, "e"] & G_V \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & H_W \end{tikzcd}$$

Let us say that we have conservative $d : D \to G$ with $fd = gd$, I need a unique conservative $u : L_D \to K_E$ that makes $eu = d$.

$$\begin{tikzcd} D \arrow[d, "u"'] \arrow[dr, "d"] & & \\ E \arrow[r, "e"] & V \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & W \end{tikzcd}$$

u comes from the fact that the category of sets has equalisers. We just need to check that it is conservative. If X is winning in K_E then $u^{-1}(X) = d^{-1}e(X)$ is winning in L_D.

$\square$

Theorem 40. **A** *also has all coequalisers.*

Proof. The proof follows the same pattern - establishing the coequalizer from the category of sets. Choose the set of winning coalitions to be as large as possible to make the arrow the coequaliser conservative.

□

Comments 41. The coproduct of two games (G and H) has as few voters as possible subject to the constraint that it has a unique arrow u to satisfy the commutative diagrams on the sets of voters. It also has as many coalitions as possible subject to the constraint that the maps from G and H are conservative.

The equaliser of two games has as many voters as possible subject to the constraint that it has a unique arrow u to satisfy the commutative diagrams on the sets of voters. It has as few winning coalitions as possible subject to the constraint that the equaliser arrow is conservative.

In general, limits have as many voters as possible and as few winning coalitions as possible subject to constraints.

Colimits have as few voters as possible and as many winning coalitions as possible subject to constraints.

Theorem 42. **A** *has all finite limits and colimits.*

Proof. The fact that we have all finite limits is immediate from the fact that we have terminal objects, finite products and equalisers and [4, Theorem 4.11]

□

Comments 43. As we know, the category of groups is left-adjoint to the category of sets via the mappings that send each set to the free group it generates and each group to its underlying set. Do we have an analogue of the free group here? Two options suggest themselves: the game that always wins and the game that always loses. It turns out that these deliver a right and left adjunction to the category of sets. In turn, this gives us a better understanding of limits and colimits in the category of SVGs

Theorem 44. *Let* $\mathbf{S}$ *be the category of sets.* $\mathbf{S} \dashv \mathbf{A}$

If required, this notation is explained in [4, Chapter 10]

Proof. First I need to define functors P and Q between $\mathbf{S}$ and $\mathbf{A}$

P is the functor that takes:

1. every set to the game, with those voters, that always wins
2. every function $(f : S \to T)$ on sets to the conservative arrow that maps the voters of $P(S)$ to the voters of $P(T)$. We know that it is conservative because the domain contains all winning coalitions

Q is the functor that takes

1. every game to the set of its voters
2. every arrow between games to the underlying function on the sets of voters

It would be typical to refer to Q as a 'forgetful' functor because it forgets the game structure.

$$\begin{tikzcd} \mathbf{S} \arrow[r, shift left, "P"] & \mathbf{A} \arrow[l, shift left, "Q"] \end{tikzcd}$$

The counit ξ needs to be a natural transformation $\xi : PQ \to 1_A$. The obvious choice is the identity on voters and coalitions (of course this may not be the identity arrow on games as the domain is usually a bigger game than the codomain). This is always conservative because $PQ(G)$ is the game, with the same voters as G, that always wins. I need to show that this arrow is universal. I need a one-to-one and onto correspondence between arrows (g) from $P(S)$ to G and arrows (f) from S to $Q(G)$ that make the following triangle commute.

$$
\begin{tikzcd}
Q(G) & PQ(G) \arrow[r, "\xi_G"] & G \\
S \arrow[u, "f"'] & P(S) \arrow[u, "P(f)"'] \arrow[ur, "g"'] &
\end{tikzcd}
$$

This is not too difficult, given g, f is just the corresponding function on the voters. Given f, there is always a corresponding, conservative g because $P(S)$ is the game with all winning coalitions.

The unit η needs to be a natural transformation $\eta : 1_S \to QP$. For all S, η_S is simply the identity - we add all winning coalitions to the set and then strip them off again. Q is the left inverse of P. Of course P is not iso because Q is not a right inverse for P. Again, I need to show that η is universal. I need a one-to-one and onto correspondence between arrows (f) from S to $Q(G)$ and arrows (g) from $P(S)$ to G that make the following triangle commute.

$$
\begin{tikzcd}
S \arrow[r, "\eta_S"] \arrow[dr, "f"'] & QP(S) \arrow[d, "Q(g)"] & P(S) \arrow[d, "g"] \\
 & Q(G) & G
\end{tikzcd}
$$

Given f, a function on sets, g is the corresponding function on games that maps the voters of $P(S)$ to the voters of G under f. We know that it is conservative because $P(S)$ has all winning coalitions.

For completeness, I check the triangle identities. I need to show that the following triangle commutes.

$$
\begin{tikzcd}
PQP(S) \arrow[r, "\xi_{P(S)}"] & P(S) \\
P(S) \arrow[u, "P(\eta_S)"] \arrow[ur, "1_{P(S)}"'] &
\end{tikzcd}
$$

η_S is the identity on S. $P(\eta_S)$ is the corresponding identity on the game that always wins. ξ is the identity on voters and coalitions so certainly these combine to make the identity on the game that always wins. Also, the following diagram commutes:

$$\begin{tikzcd}
QPQ(G) \arrow[r, "Q(\xi_G)"] & Q(G) \\
Q(G) \arrow[u, "\eta_{Q(G)}"] \arrow[ur, "1_{Q(G)}"'] &
\end{tikzcd}$$

$\eta_{Q(G)}$ is the identity on the set of voters of G. ξ_G is the mapping between games which is the identity on the voters (but need not be on coalitions). Q just strips off the coalition structure so $Q(\xi_G)$ is just the identity on sets. These two combine to give the identity on sets.

□

Theorem 45. *Let* **S** *be the category of sets.* $\mathbf{A} \dashv \mathbf{S}$

Proof. The proof is very simliar to that of Theorem 44.

First I need to define functors P and Q between **A** and **S**.

Q is the functor that takes

1. every set to the game, with those voters, that always loses
2. every function $(f : S \to T)$ on sets to the conservative arrow that maps the voters of $Q(S)$ to the voters of $Q(T)$. We know that it is conservative because the codomain contains no winning coalitions

P is the functor that takes

1. every game to the set of its voters

2. every arrow between games to the underlying function on the sets of voters

It would be typical to refer to P as a 'forgetful' functor because it forgets the game structure.

$$\mathbf{A} \underset{Q}{\overset{P}{\rightleftarrows}} \mathbf{S}$$

The counit ξ needs to be a natural transformation $\xi : PQ \to 1_S$. The obvious choice is the identity. I need to show that this arrow is universal. I need a one-to-one and onto correspondence between arrows (g) from $P(G)$ to S and arrows (f) from G to $Q(S)$ that make the following triangle commute.

$$\begin{tikzcd}
Q(S) & PQ(S) \arrow[r, "\xi_S"] & S \\
G \arrow[u, "f"'] & P(G) \arrow[u, "P(f)"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

Given a function g from the voters of G to S, we can extend this into an f from G to the game $Q(S)$. This must be conservative because there are no winning coalitions in the codomain. In the other direction, given f, the corresponding g is just the function between the voters.

The unit η needs to be a natural transformation $\eta : 1_G \to QP$. The obvious choice for η_G is the arrow that is the identity on voters. This is conservative as the codomain is the game with no winning coalitions. I need to show that η is universal. I need a one-to-one and onto correspondence between arrows (f) from G to $Q(S)$ and arrows (g) from $P(G)$ to S that make the following triangle commute.

$$\begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[dr, "f"'] & QP(G) \arrow[d, "Q(g)"] & P(G) \arrow[d, "g"] \\ & Q(S) & S \end{tikzcd}$$

Given f, an arrow between games, g is the corresponding function on the voters. If we start with g as a function of sets then we know that we can build an arrow f with g as the function on the voters. The lack of winning coalitions in the codomain of f ($Q(S)$) means that we know that the arrow will be conservative.

Finally, the triangle identities:

$$\begin{tikzcd} PQP(G) \arrow[r, "\xi_{P(G)}"] & P(G) \\ P(G) \arrow[u, "P(\eta_G)"] \arrow[ur, "1_{P(G)}"'] & \end{tikzcd}$$

η_G is the arrow on games that is the identity on the voters of G. $P(\eta_G)$ is just the identity on the voters as a set. $\xi_{P(G)}$ is the identity on the voters.

$$\begin{tikzcd}
QPQ(S) \arrow[r, "Q(\xi_S)"] & Q(S) \\
Q(S) \arrow[u, "\eta_{Q(S)}"] \arrow[ur, "1_{Q(S)}"'] &
\end{tikzcd}$$

$\eta_{Q(S)}$ is the identity on voters mapping from the game that always loses to itself. ξ_S is the identity on the set S and $Q(\xi_S)$ is the identity on the voters from the game that always loses to the game that always loses.

□

In both cases, the mapping from games to sets just strips off the winning coalitions to leave the set of voters. But we have two equivalents of forming the 'free group': one maps a set of voters to the game that always wins on those voters and one maps a set of voters to the game that always loses.

The functor, from **S** to **A**, which adds all winning coalitions is left adjunct to the forgetful functor. We know that the right adjunct functor (the forgetful functor) preserves limits. This tells us that the product of two objects in **A** (G and H) will need to have, as voters, the cartesian product of the voter sets of G and H. It also tells us that a terminal object in **A** will have to have, as voters, the terminal object in **S** i.e. the one-element set. This is what we see.

The forgetful functor from **A** to **S** is left adjunct to the functor that maps each set to the game with no winning coalitions. We know that the left adjunct functor preserves colimits. This tells us that the coproduct of two objects in $\mathbf{A}$ (G and H) will need to have, as voters, the direct sum of the voter sets of G and H. It also tells us that an initial object in $\mathbf{A}$ will have to have, as voters, the initial object in $\mathbf{S}$ i.e. the empty set. This is what we see.

$\mathbf{A} \dashv \mathbf{S}$ and $\mathbf{S} \dashv \mathbf{A}$ tell us that $\mathbf{A} \dashv \mathbf{A}$.

Comments 46. So we have a category of simple voting games. On its own, it does not take our understanding of SVGs much further forward. On the other hand it will link together other things in the paper.

What is wrong with this approach? Well it could be that it is a bit set-theoretic in its approach. Saunders Mac Lane once described category theory as 'getting by without elements' but here the voters play a big role; we haven't really got by without elements.

As an antidote to that, we tried a new category, one in which the focus was not on voters but, instead, on bipartitions.

3 The Category of Bipartitions

Definition 47. Given a set V of voters, a *bipartition* of V is a function from V to $\{\bot, \top\}$. This breaks V into two camps: those who vote 'yes' and those who vote 'no'. The definition is taken from [1, Definition 2.1.5].

Comments 48. It is possible to think of every bipartition as a subset of V (the subset of those that voted 'yes'). One could also think of it as the subset that voted 'no'. The reason for using the terminology of bipartitions is its symmetric nature. We will see that this is beneficial when we come to think about duality and simplicial complexes.

Definition 49. For a given V, the set of bipartitions has a natural partial order. If D and E are bipartitions then $D \geq E$ iff, for all $v \in V, E(v) = \top \implies D(v) = \top$

Definition 50. For any finite set V, let $B(V)$ be the set of all bipartitions of V. A function, f from $B(V)$ to $B(W)$ is *monotonic* if $D \leq E \implies f(D) \leq f(E)$ for all D and E in $B(V)$.

Comments 51. Every mapping of the voters $f : V \to W$ defines a monotonic map from $B(V)$ to $B(W)$ but there are many monotonic maps that do not correspond to mappings of the voters.

Definition 52. $\mathbf{B}$ is the category with the $B(V)$ as objects and monotonic maps as arrows. I will refer to this as the *Category of Bipartitions.*

Comments 53. This is an interesting avenue of enquiry because we can think of every SVG as a monotonic map from the set of bipartitions to the ordered two object set ($B(\{1\})$ can play this role). This suggests the possibility of expressing the category of SVGs as the slice of bipartitions over $B(\{1\})$.

Comments 54. This is potentially attractive because we have an adjunction [4, P101]:

$$\mathbf{B}/B(\{1\}) \underset{B^*}{\overset{\Sigma_B}{\rightleftarrows}} \mathbf{B}$$

Where Σ_B just returns the domain of the arrow and B^* maps each object $B(\{A\})$ to the projection arrow $p_1 : B(\{1\}) \times B(\{A\}) \to B(\{1\})$. This means that limits in $\mathbf{B}$ correspond to limits in $\mathbf{B}/B(\{1\})$ via B^*.

Also, like all slice categories [4, p100] $\mathbf{B}/B(\{1\})$ has a terminal object: $1_{B(\{1\})}$.

Comments 55. So what are the properties of $\mathbf{B}$?

[4, Exercise 10.16] makes the point that, for a Cartesian Closed Category $\mathbf{C}$ (in this case, the category of finite sets) and an object A (in this case, a set with two objects) of $\mathbf{C}$, the operation of taking objects and arrows in $\mathbf{C}$ and raising A to the power of them. is a contravariant functor: $A^- : \mathbf{C} \to \mathbf{C^{op}}$ which is self adjoint. This means that the image, under A^-, in the category of finite sets, of a colimit is a colimit in $\mathbf{C^{op}}$ or a limit in $\mathbf{C}$. In the context of power sets, this gives us the familiar identity: $2^{A+B} = 2^A \times 2^B$. That is to say that every subset of the direct sum (if A and B have objects in common then two copies of these are placed in the direct sum) of A and B can be thought of as an ordered pair of a subset of A and a subset of B. Or, to put it another way, any function from the direct sum of two sets can be thought of as an ordered pair of two functions, one from each of the sets.

This would lead us to

Theorem 56. $\mathbf{B}$ *has all products. In particular, the product of* $B(V)$ *and* $B(W)$ *is the following diagram.*

$$B(V) \xleftarrow[p_1]{} B(V+W) \xrightarrow[p_2]{} B(W)$$

Where p_1 *creates a bipartition of* V *from a bipartition of* $V+W$ *by simply asking where the elements of* V *were mapped to in the bipartition of* $V+W$*. It is just a restriction of the function.*

Proof. $V+W$ is the direct sum of V and W so

$$
\begin{tikzcd}
V \arrow[r, "q_1"'] & V+W & W \arrow[l, "q_2"]
\end{tikzcd}
$$

Is a coproduct diagram.

Let us say we have arrows $s : B(T) \to B(V)$ and $t : B(T) \to B(W)$. We can display these in the following diagram.

$$
\begin{tikzcd}
 & B(T) \arrow[dl, "s"] \arrow[d, "u"'] \arrow[dr, "t"'] & \\
B(V) & B(V+W) \arrow[l, "p_1"] \arrow[r, "p_2"'] & B(W)
\end{tikzcd}
$$

We need a unique u to make the diagram commute.

How would we build such a u? For every object of $B(T)$, we have a corresponding bipartition of V and a bipartition of W. So we define the corresponding bipartition of $V+W$ to look like the bipartition of V on the copy of V and the biparition of W on the copy of W. This gives us a unique biparition of $V+W$.

To be more precise, $B(T)$ consists of mappings $f : T \to \{\perp, \top\}$. so $s(f)$ and $t(f)$ are mappings from V and W respectively to $\{\perp, \top\}$. This gives us the following diagram.

$$
\begin{tikzcd}
 & \{\perp, \top\} & \\
V \arrow[ur, "s(f)"] \arrow[r, "q_1"'] & V+W \arrow[u] & W \arrow[ul, "t(f)"'] \arrow[l, "q_2"]
\end{tikzcd}
$$

Since

$$
\begin{tikzcd}
V \arrow[r, "q_1"'] & V+W & W \arrow[l, "q_2"]
\end{tikzcd}
$$

Is a coproduct, there must be a unique $u(f) : V+W \to \{\perp, \top\}$ that makes the diagram commute. This member of $B(V+W)$ is the value of u at $f \in B(T)$.

Finally, we need to show that u is monotonic.

To obtain a contradiction, let us assume $f \leq g$ and $\neg(u(f) \leq u(g))$.

In this case, there must be $t \in V+W$ such that $u(f)(t) = \top$ and $u(g)(t) = \perp$.

Since $t \in V+W$, it must be the case that $t \in V$ or $t \in W$. Let us assume WLOG that $t \in V$

In this case $q_1 s(f)(t) = \top$ and $q_1 s(g)(t) = \perp$. Since q_1 is the inclusion function.

$s(f)(t) = \top$ and $s(g)(t) = \perp$. This is not possible as s, as an arrow, must be monotonic and $f \leq g$ so it must be the case that $s(f) \leq s(g)$.

$\square$

Theorem 57. *$B(\emptyset)$ is terminal in* $\mathbf{B}$.

Proof. There is only one element of $B(\emptyset)$, the unique empty function from the empty set to $\{\bot, \top\}$.

For any set A, there is a unique map from $B(A)$ to $B(\emptyset)$: the map that takes every function from A to $\{\bot, \top\}$ to the unique function from the empty set to $\{\bot, \top\}$ $(! : \emptyset \to \{\bot, \top\})$. This is monotonic vacuously; it can never be the case that $!(f)(v) = \top$ and $!(g)(w) = \bot$.

$\square$

However, there is a problem

Theorem 58. $\mathbf{B}$ *does not contain all pullbacks.*

Proof. There are some pullbacks.

When the corner of arrows looks like this.

$$\begin{tikzcd}
 & B(X) \arrow[d, "\{\bot, \top\}^f"] \\
B(W) \arrow[r, "\{\bot, \top\}^g"'] & B(V)
\end{tikzcd}$$

Then we have the following in the category of finite sets.

$$\begin{tikzcd}
 & X \\
W & V \arrow[l, "g"] \arrow[u, "f"']
\end{tikzcd}$$

The category of sets contains all pushouts and so we have this

$$\begin{tikzcd}
Z & X \arrow[l, "q_1"] \\
W \arrow[u, "q_2"'] & V \arrow[l, "g"] \arrow[u, "f"']
\end{tikzcd}$$

Which, through the adjunction gives us this pullback:

$$
\begin{tikzcd}
B(Z) \arrow[r, "\{\bot,\top\}^{q_1}"'] \arrow[d, "\{\bot,\top\}^{q_2}"] & B(X) \arrow[d, "\{\bot,\top\}^{f}"] \\
B(W) \arrow[r, "\{\bot,\top\}^{g}"'] & B(V)
\end{tikzcd}
$$

However, we have other corners of arrows where the arrows as not of the form $\{\bot, \top\}^f$ i.e. they can't be expressed as mappings on the voters. Some of these do not have a pullback. Since every pullback can be thought of as an equaliser ([4, Theorem 4.5], to show this, all we need is a pair of parallel arrows that do not have an equaliser.

The problem with equalisers also comes with arrows that are not of this form. For example let $f : B(\{1,2\}) \to B(\{1,2\})$ be the monotonic function that is the identity except that it maps $\{1\}$ to $\{1,2\}$ and let the other arrow in the parallel pair be the identity on $B(\{1,2\})$. Equalisers need to be monic ([4, Theorem 2.6]) and in this category, an arrow can only be monic if it is one-to-one. (This is really easy to show iff $f(v) = f(w)$ with $v \neq w$, $g(x)$ is the constant function equal to v and $h(x)$ is the constant function equal to w then $fg = fh$ but $g \neq h$). There can be no one-to-one equaliser because the cardinality of the domain must be a power of 2 (because it is a bipartition object) and, in this case, we would need the cardinality of the domain to be 3.

$\square$

Definition 59. I define $\mathbf{D}$ to be $\mathbf{B}/B(\{1\}$. As described in [4, Chapter 11], that is, the category which has, as objects, all arrows $f : B(V) \to B(\{1\})$ from objects of $\mathbf{B}$ to $B(\{1\})$. The arrows are arrows $h : B(V) \to B(W)$ of $\mathbf{B}$ such that the following triangle commutes.

$$\begin{tikzcd} B(V) \arrow[rr, "h"] \arrow[dr, "f"'] & & B(W) \arrow[dl, "g"] \\ & B(\{1\}) & \end{tikzcd}$$

In $\mathbf{D}$ we think of h as an arrow with domain f and codomain g.

Comments 60. $\mathbf{D}$ has a terminal object $1_{B\{1\}} : B\{1\} \to B\{1\}$ but it doesn't even contain products because $\mathbf{B}$ doesn't have pullbacks.

There is another problem. This category does contain all SVGs as objects but it is hard to attach a real-world meaning to arrows that cannot be expressed as $\{\perp, \top\}^f$.

These categories will play a role later on but on their own they are not the answer.

4 Lattice Categories of Simple Voting Games

4.1 Defining the Lattices

Definition 61. *The category T* has two objects:$\perp$ and $\top$ and one non-identity arrow from $\perp$ to $\top$.

Comments 62. T can be thought of as the category of simple voting games (SVGs) with no voters. Of course there are two of these (both 'degenerate'; they would not be allowed by Definition 1). $\top$ is the game that passes every bill and $\perp$ is the game that does not pass any bill. These two games are not commonly used in the theory of simple voting games. In real world applications they are not very important. In this work we will see that they are essential.

Definition 63. For each integer n, let C_n be defined, by recursion, as the category of functors from $T \to C_{n-1}$.

$C_0 = T$. If A and B are objects of C_{n-1}, I write:

1. a functor that maps $\perp$ to A and $\top$ to B as $[A, B]$.

2. the natural transformation from $[A, B]$ to $[D, E]$, with components $f : A \to D$ and $g : B \to E$ as $[f, g]$.

3. the identity arrow from A to A as 1_A.

I use capital letters (A, B, D, E, F, G) from the beginning of the alphabet to represent members of the C_n.

Comments 64. It will become clear that C_n is the category of SVGs with n voters. We will see that the move from C_n to C_{n+1} involves the addition of a new voter (v_{n+1}). A is the SVG with n voters that results if v_{n+1} votes 'no' in $[A, B]$. B is the SVG with n voters that results if v_{n+1} votes 'yes' in $[A, B]$. This is one way of interpreting objects in C_n as SVGs. Later, we will find others. An arrow from A to B says that A is smaller than B. We can also use this arrow to represent the SVG that turns into A if the n^{th} voter votes 'no' and B if the n^{th} voter votes 'yes'.

Comments 65. Each of the C_n is a partially ordered set. The following notation will reinforce this. It would have been possible to do this part of the work without using the language of category theory. However, it feels more in the spirit of our project to do so. [2, Page 9 (Example 8)] shows that every partially ordered set can be expressed as a category and every category with no more than one arrow between each pair of objects corresponds to a partially ordered set.

Definition 66. If A and B are objects of C_n then I say that $A \leq B$ iff there is an arrow from A to B.

Comments 67. When we think of A and B as SVGs, $A \leq B$ means that, for a given division, if the SVG A passes the bill then the SVG B will also pass the bill; any bipartition ([1, Definition 2.1.5]) that passes the bill under A will also pass the bill under B. A shorthand for this is to say that B is bigger than or equal to A. As a first step, we can see that $\leq$ corresponds to this meaning in T.

Theorem 68. *Let A and B be objects of C_{n-1}. $[A, B]$ is an object of C_n if and only if $A \leq B$.*

Proof. $[A, B]$ is an object in the category of functors from T to C_{n-1}. The functor maps $\perp$ to A and $\top$ to B therefore, it must map the arrow from $\perp$ to $\top$ to an arrow from A to B and so, by Definition 66, $A \leq B$.

Say $A \leq B$ then we can build a functor that maps $\perp$ to A, $\top$ to B and the arrow from $\perp$ to $\top$ to the arrow from A to B. This is $[A, B]$.

□

Comments 69. This says that monotonicity (as defined in [1, Definition 2.1.1.3]) is built into games from the beginning. That is to say that if the n^{th}

voter switches from 'no' to 'yes' the resulting game will always be at least as permissive. It also says that every possible monotone game is in C_n.

Theorem 70. *If $[A, B]$ and $[D, E]$ are objects of C_n then $[A, B] \leq [D, E]$ iff $A \leq D$ and $B \leq E$.*

Proof. Let us assume that $[A, B] \leq [D, E]$. Then (By Definition 66) there is an arrow $f : [A, B] \to [D, E]$. C_n is a functor category and so this f is a natural transformation with two components: $f_\perp : A \to D$ and $f_\top : B \to E$. This tells us that $A \leq D$ and $B \leq E$.

If $A \leq D$ and $B \leq E$ then (By Definition 66) we have $f : A \to D$ and $g : B \to E$. From these arrows, we can build a natural transformation $[f, g] : [A, B] \to [D, E]$ showing that $[A, B] \leq [D, E]$.

□

Theorem 71. *There is at most one arrow from one object of C_n to another.*

Proof. The proof is by induction.

The result is true in C_0.

Let us assume that it is true in C_r.

Consider two objects of C_{r+1}: $[A, B]$ and $[D, E]$. To obtain a contradiction, let us assume that there are two arrows from $[A, B]$ to $[D, E]$. This, Theorem 70 and Definition 66 implies that there must be two arrows from A to D or from B to E. This cannot be true, by the induction hypothesis.

□

Theorem 72. *For every object $[A, B]$ of C_n,* the dual of $[A, B]$: $[A, B]^*$ *(defined below), is also an object of C_n.*

At the same time, we will prove that $[A, B] \leq [D, E] \iff [A, B]^* \geq [D, E]^*$.

So $[A, B]^*$ *is defined as follows:*

In C_0, $\perp^* = \top$ *and* $\top^* = \perp$

In C_n, *for* $n \geq 1$, $[A, B]^* = [B^*, A^*]$

Proof. The proof is by induction. For $n = 0$, it is obvious, from the definition, that G^* is in C_0 for G equal to $\top$ and $\perp$. Also

if $A = \perp$ and $B = \perp$ then $\perp \leq \perp$ and $\top \geq \top$

if $A = \perp$ and $B = \top$ then $\perp \leq \top$ and $\top \geq \perp$

if $A = \top$ and $B = \perp$ then $\neg(\top \leq \perp)$ and $\neg(\perp \geq \top)$

if $A = \top$ and $B = \top$ then $\top \leq \top$ and $\perp \geq \perp$

Let us assume that we have both results for $n = k$ so for every G in C_k, G^* is in C_k and for G and H objects of C_k, $G \leq H \iff H^* \leq G^*$.

Now let $[A, B]$ be an object of C_{k+1}.

We know that $A \leq B$ (Theorem 68) where A and B are in C_k.

And so $B^* \leq A^*$ (By the induction hypothesis on the inequality).

This, and Theorem 68, imply that $[B^*, A^*]$ is an object of C_{k+1}.

Since $[B^*, A^*] = [A, B]^*$, we have shown that $[A, B]^*$ is a member of C_{k+1}. So we have established the induction on the existence of the dual.

Next, we need to carry out the induction step for the inequality.

Let us assume that A, B, D and E are objects of C_k.

$[A, B] \leq [D, E]$

$\iff A \leq D$ and $B \leq E$ (Theorem 70)

$\iff D^* \leq A^*$ and $E^* \leq B^*$ (Induction Hypothesis)

$\iff [E^*, D^*] \leq [B^*, A^*]$ (Theorem 70)

$\iff [D, E]^* \leq [A, B]^*$ (Definition of dual).

And we have completed the induction step.

$\square$

Comments 73. The idea of duality is natural and important. It was hardly mentioned and certainly not focussed on until the publication of [1]; since then, other writers have also made a great deal of use of duality (For example [3]). The dual of $\mathcal{G}$ is the game that gives the same real-world outcome as $\mathcal{G}$ when the issue to be decided is negated and all the voters reverse their votes. It is natural to believe that the two would cancel out but this is only true in self-dual games (e.g. one in which everybody has an equal vote and the quota is half of the total weight).

For example. Let us imagine a country of three people which required unanimity from the population to raise taxes. We will show that if the bill was reversed (taxes will not be raised) and everybody in the population consequently reversed their votes then the decision rule that would give the matching outcome is the dual: the game which wins if at least one person votes 'yes'.

Let us examine this. First, let us assume that all three people want to raise taxes. So the referendum asks 'do you want to not raise taxes?'. They all say 'no'. Therefore the rule does not pass and taxes are raised. Which matches what we would have got under the other method.

If two out of three people wanted to raise taxes, then we would have got one 'yes'. Thus the motion would pass and taxes would not be raised.

If nobody wanted to raise taxes then they would all say 'yes'. The motion would pass (easily! Just as in the original game, the motion would be far from passing) and taxes would not be raised.

Forming the dual requires taking two complements. The first is a complement relative to the power set of V (this corresponds to negating the question) and the second corresponds to taking the complement of each coalition relative to V (this corresponds to the voters reversing their view). Carrying out these two operations leaves us with an SVG; carrying out just one would not. The resulting hypergraph would not be monotonic.

Definition 74. If $f : V \to \{\perp, \top\}$ is a bipartition then the *opposite of* f, $\neg f$ is the map defined $\neg f(v) = \perp$ when $f(v) = \top$ and $\neg f(v) = \top$ when $f(v) = \perp$

Theorem 75. *$(G^*)^* = G$ for all objects G in C_n.*

Proof. The proof is by induction on n. It is clearly true for $\perp$ and $\top$ in C_0 (By Theorem 72). $\top^* = \perp$ and $\perp^* = \top$.

Let us assume that the theorem is true for objects of C_k

Let $A = [B, D]$ be an object of C_{k+1}.

$([B, D]^*)^*$

$= [D^*, B^*]^*$ (Using Theorem 72)

$= [(B^*)^*, (D^*)^*]$ (Using Theorem 72)

$= [B, D]$ (By the induction hypothesis)

$= A$.

□

Comments 76. The dual of A will turn out to be the game whose bipartitions that have positive outcome are such that their opposites are the blocking bipartitions of A. That is, they are the opposites of the bipartitions that have negative outcome under A. Theorem 75 gives us justification for calling it a dual.

Theorem 77. *We can extend the function of taking-the-dual to a contravarient functor on C_n.*

Proof. There is an arrow from A to B

$\implies A \leq B$ (Definition 66)

$\implies B^* \leq A^*$ (Theorem 72)

$\implies$ There is an arrow from B^* to A^*. (Definition 66)

This arrow is the image of the arrow from A to B under the functor. Theorem 71 tells us that there can only be one such arrow.

□

Definition 78. Let $f : A \to B$ be an arrow of C_n. I will refer to the image of f under the dual contravarient functor as the *dual arrow* $f^* : B^* \to A^*$.

Theorem 79. *If $f : A \to B$ and $g : D \to E$ are arrows of C_n then*

$[f, g]^* = [g^*, f^*]$.

Proof. By Definition 63, $[f, g]$ is the arrow from $[A, D]$ to $[B, E]$.

By Definition 78, $[f, g]^*$ is the arrow from $[B, E]^*$ to $[A, D]^*$.

Using Definition 72, $[f, g]^*$ is the arrow from $[E^*, B^*]$ to $[D^*, A^*]$

Using Definition 78, the arrow from $[E^*, B^*]$ to $[D^*, A^*]$ is $[g^*, f^*]$.

So $[f, g]^* = [g^*, f^*]$.

□

Theorem 80. *The dual mapping is one-to-one and onto; it is a bijective functor.*

Proof. This is proved by the fact that * has a two-sided inverse: itself (Theorem 75).

□

Theorem 81. *If A and B are objects of C_n then $A \leq B$ and $B \leq A$ imply $A = B$.*

Proof. The proof is by induction. It is true in C_0. If $A \leq B$ and $B \leq A$ then (using Definition 66) there are arrows from A to B and from B to A. In C_0 this can only mean that $A = B = \perp$ or $A = B = \top$ and the arrows are the identity.

Let us assume that the result is true in C_r. Let $[A, B]$ and $[D, E]$ be objects of C_{r+1}. Then if $[A, B] \leq [D, E]$ and $[D, E] \leq [A, B]$ we know that $A \leq D$; $B \leq E$; $D \leq A$ and $E \leq B$ (By Theorem 70). By the induction hypothesis, this tells us that $A = D$ and $B = E$ and so $[A, B] = [D, E]$.

□

4.2 Five Functors Between the Lattice Categories

Comments 82. I now define three injections of C_n into C_{n+1} and two surjections from C_n to C_{n-1}.

Definition 83. The *dummy functor,* $\mathit{Dum}_n : C_n \to C_{n+1}$ maps A, an object of C_n, to the constant functor $\mathit{Dum}_n(A) : T \to C_n$ which is equal to A for both objects of T. It maps arrows $(f : A \to B)$ of C_n to the natural transformation from $\mathit{Dum}_n(A)$ to $\mathit{Dum}_n(B)$ which has f as both components.

In short $\mathit{Dum}_n(A) = [A, A]$ and $\mathit{Dum}_n(f) = [f, f]$.

Comments 84. Dum_n maps a game with n voters to a game with $n + 1$ voters. This consists of the original game with v_{n+1} added as a dummy.

Definition 85. I define $\perp_n$ and $\top_n$ in C_n by recursion for all non-negative integers n.

C_0 has two objects: $\perp$ and $\top$ (Definition 61). I will also call these $\perp_0$ *and* $\top_0$.

In C_n, I define $\perp_n = Dum_{n-1}(\perp_{n-1})$ and $\top_n = Dum_{n-1}(\top_{n-1})$.

Comments 86. If we add a dummy to the game that always passes or the game that always blocks then the result is the game that always passes or the game that always blocks, respectively.

Theorem 87. *For $\perp_n$ and $\top_n$ in C_n, $\perp_n^* = \top_n$ and $\top_n^* = \perp_n$.*

Proof. The proof is by induction. Theorem 72 tells us that the theorem is true in C_0.

Let us assume that it holds in C_k.

$(\perp_{k+1})^*$

$= [\perp_k, \perp_k]^*$ (Using Definition 85.)

$= [(\perp_k)^*, (\perp_k)^*]$ (Using Theorem 72.)

$= [\top_k, \top_k]$ (By the Induction Hypothesis.)

$= \top_{k+1}$ (Using Definition 85.)

The fact that $(\top_{k+1})^* = \perp_{k+1}$ follows by duality.

□

Definition 88. For each n, there is a *veto functor,* $Vet_n : C_n \to C_{n+1}$, which maps each object, A, of C_n, to the functor $Vet_n(A) : T \to C_n$. Vet_n maps $\perp_0$ to $\perp_n$ and $\top_0$ to A. It maps arrows $(f : A \to B)$ of C_n to the natural transformation from $Vet_n(A)$ to $Vet_n(B)$ which has the component $1_{\perp_n}$ corresponding to $\perp_0$ and f corresponding to $\top_0$.

In summary $Vet_n(A) = [\perp_n, A]$ and $Vet_n(f) = [1_{\perp_n}, f]$.

Comments 89. Vet_n takes the existing game, in C_n and adds a new voter (v_{n+1}) as a vetoer. That is to say that if the new voter votes 'no' then the bill will not pass under the game. If the new voter votes 'yes' then the decision will revert to the game that existed before Vet_n was applied.

Definition 90. For each n, there is a *Passer functor,* $Pas_n : C_n \to C_{n+1}$ which maps A, an object of C_n, to the functor $Pas_n(A) : T \to C_n$. $Pas_n(A)$ maps $\perp_0$ to A and $\top_0$ to $\top_n$. It maps arrows $(f : A \to B)$ of C_n to the natural transformation from $Pas_n(A)$ to $Pas_n(B)$ which has the component f corresponding to $\perp_0$ and $1_{\top_n}$ corresponding to $\top_0$.

In summary $Pas_n(A) = [A, \top_n]$ and $Pas_n(f) = [f, 1_{\top_n}]$.

Comments 91. Pas_n takes the existing game, in C_n and adds a new voter (v_n) as a passer. That is to say that if the new voter votes 'yes' then the bill will pass under the game. If the new voter votes 'no' then the decision will revert to the game that existed before Pas_n was applied.

Definition 92. For each $n \geq 1$ there is a *domain functor*: $Dom_n : C_n \to C_{n-1}$. Dom_n maps $[A, B]$, an object of C_n, to the object A of C_{n-1}. It maps the arrow $[f, g]$ of C_n to the arrow f of C_{n-1}.

Comments 93. Dom_n takes the existing game and assumes that v_n votes 'no'. What results is a game for the remaining $n-1$ voters.

Definition 94. For each n there is a *codomain functor*: $Cod_n : C_n \to C_{n-1}$. Cod_n maps $[A, B]$, an object of C_n to the object B of C_{n-1}. It maps the arrow $[f, g]$ of C_n to the arrow g of C_{n-1}.

Comments 95. Cod_n takes the existing game and assumes that v_n votes 'yes'. What results is a game for the remaining $n-1$ voters.

Comments 96. Four of these functors are dual to each other in pairs. Dum_n is self-dual. So effectively we have three functors. None of them look particularly complex or remarkable. It is surprising how much we can do with them.

Theorem 97. *Let A be an object of C_n. $Vet_n(A^*) = Pas_n(A)^*$.*

Let f be an arrow of C_n. $Vet_n(f^*) = Pas_n(f)^*$

Proof. $Vet_n(A^*) = Pas_n(A)^*$.

$\iff [\perp_{n-1}, A^*] = [A, \top_{n-1}]^*$ (Using Definition 88 and Definition 90)

$\iff [\perp_{n-1}, A^*] = [\top^*_{n-1}, A^*]$ (Using Theorem 72)

$\iff [\perp_{n-1}, A^*] = [\perp_{n-1}, A^*]$ (Using Theorem 87)

$Vet_n(f^*) = Pas_n(f)^*$.

$\iff [1_{\perp_{n-1}}, f^*] = [f, 1_{\top_{n-1}}]^*$ (Using Definition 88 and Definition 90)

$\iff [1_{\perp_{n-1}}, f^*] = [1^*_{\top_{n-1}}, f^*]$ (Using Theorem 79)

$\iff [1_{\perp_{n-1}}, f^*] = [1_{\perp_{n-1}}, f^*]$ (Using Definition 78)

□

Theorem 98. *Let A be an object of C_n. $Vet_n(A)^* = Pas_n(A^*)$.*

Proof. By duality.

□

Theorem 99. *Let A be an object of C_n.*

$Dom_n(A^*) = Cod_n(A)^*$.

Proof. Let $A = [D, E]$.

$Dom_n([D, E]^*) = Cod_n([D, E])^*$

$\iff Dom_n([E^*, D^*]) = E^*$ (Using Theorem 72 and Definition 94)

$\iff E^* = E^*$ (Using Definition 92).

□

Theorem 100. *Let A be an object of C_n. $Cod_n(A^*) = Dom_n(A)^*$*

Proof. By duality.

□

Theorem 101. *$Dum_n : C_{n-1} \to C_n$ is self-dual; $Dum_n(A^*) = Dum_n(A)^*$*

Proof. $Dum_n(A^*)$

$= [A^*, A^*]$ (Using Definition 83)

$= [A, A]^*$ (Using Theorem 72)

$= Dum_n(A)^*$ (Using Definition 83).

□

Theorem 102. *$\perp_n$ is initial in C_n.*

Proof. The proof is by induction on n.

It is true in C_0. There is one arrow from $\perp$ to itself ($1_\perp$). There is also one arrow from $\perp$ to $\top$.

Let us assume that it is true in C_r. Let $[A, B]$ be an element of C_{r+1}. By the induction hypothesis, we have arrows from $\perp_r$ to A and B. Since C_{r+1} is the category of functors from T to C_r, this gives us an arrow from $[\perp_r, \perp_r] = \perp_{r+1}$ to $[A, B]$. By Theorem 71 there cannot be more than one arrow so there is exactly one and $\perp_{r+1}$ is initial.

□

Theorem 103. *$\top_n$ is terminal in C_n.*

Proof. By duality.

□

Theorem 104. *$\perp_n$ is minimal in C_n.*

Proof. Let A be an object of C_n. $\perp_n$ is initial (Theorem 102) and so there is an arrow from $\perp_n$ to A, hence $\perp_n \leq A$ (Definition 66).

□

Theorem 105. *$\top_n$ is maximal in C_n.*

Proof. By duality.

□

Comments 106. These theorems correspond to that not-too-astonishing fact that the game that always passes is the most permissive and the game that never passes is the least permissive.

Theorem 107. *If a functor from C_n to C_m is one-to-one on objects then it is one-to-one on arrows (faithful).*

Proof. This follows from the fact that C_n and C_m are order categories.

□

Theorem 108. *Vet_n,Pas_n, and Dum_n are one-to-one on objects and arrows. Dom_n and Cod_n are onto on objects and arrows. This means that Vet_n, Pas_n and Dum_n are faithful functors and Dom_n and Cod_n are full functors.*

Proof. $Vet_n(A) = Vet_n(B)$

$\iff [\perp_n, A] = [\perp_n, B]$ (Definition 88)

$\iff A = B$

By Theorem 107, Vet_n is also one-to-one on arrows.

By Duality, Pas_n is also one-to-one on objects and arrows.

$Dum_n(A) = Dum_n(B)$

$\iff [A, A] = [B, B]$ (Definition 83)

$\iff A = B$

By Theorem 107, Dum_n is also one-to-one on arrows.

Let A be an object of C_n

$Dom_n([A, \top_n]) = A$ (By Definition 92)

Let f be an arrow of C_n

$Dom_n([f, 1_{\top_n}]) = f$ (By Definition 92)

We know that Cod is onto by duality.

□

Theorem 109. $Vet_n(\top_n) = Pas_n(\perp_n) = [\perp_n, \top_n]$ *for all non-negative integers* n.

Proof. By Definition 88, $Vet_n(\top_n) = [\perp_n, \top_n]$.

By Definition 90, $Pas_n(\perp_n) = [\perp_n, \top_n]$.

□

Definition 110. $Dict_n$ is an object of C_n defined as $Vet_n(\top_{n-1}) = Pas_n(\perp_{n-1}) = [\perp_{n-1}, \top_{n-1}]$ where $\top_{n-1}$ and $\perp_{n-1}$ are in C_{n-1}.

Definition 111. $\wedge$ defined by recursion. In C_0, it is as follows:

$\perp_0 \wedge \perp_0 := \perp_0$

$\perp_0 \wedge \top_0 := \perp_0$

$\top_0 \wedge \perp_0 := \perp_0$

$\top_0 \wedge \top_0 := \top_0$

If A and B are objects of C_n where n is a positive integer.

$A \wedge B := [Dom_n(A) \wedge Dom_n(B), Cod_n(A) \wedge Cod_n(B)]$.

Theorem 112. *Let A and B be objects of C_n. $A \wedge B$ is the greatest lower bound of A and B.*

Proof. The proof is by induction. It is true in C_0 by Definition 111, Definition 66 and Definition 63.

Let us assume that the theorem is true for $n = k$.

Let $A = [D, E]$ and $B = [F, G]$ be objects of C_{k+1}.

$A \leq A \wedge B$

$\iff [D, E] \leq [D, E] \wedge [F, G]$

$\iff [D, E] \leq [D \wedge F, E \wedge G]$ (Definition 111)

$\iff D \leq D \wedge F$ and $E \leq E \wedge G$ (Theorem 70)

Both of these are true by the induction hypothesis and so $A \leq A \wedge B$. The proof that $B \leq A \wedge B$ is obviously similar.

Now, given H in C_n with $H \leq A$ and $H \leq B$, I need to show that $H \leq A \wedge B$.

In C_0:

If A and B are $\perp_0$ and $\perp_0$ then H must be $\perp_0$ which is $\leq A \wedge B = \perp_0 \wedge \perp_0 = \perp_0$

If A and B are $\perp_0$ and $\top_0$ respectively then H must be $\perp_0$ which is $\leq A \wedge B = \perp_0 \wedge \top_0 = \perp_0$

If A and B are $\top_0$ and $\perp_0$ respectively then H must be $\perp_0$ which is $\leq A \wedge B = \top_0 \wedge \perp_0 = \perp_0$

If A and B are $\top_0$ and $\top_0$ respectively then H could be $\perp_0$ or $\top_0$ which are both $\leq A \wedge B = \top_0 \wedge \top_0 = \top_0$

Let us assume that we have the result for $n = k$

Let $A = [D, E]$, $B = [F, G]$ and $H = [I, J]$ be objects of C_{k+1}.

$H \leq A$ and $H \leq B$

$\implies [I, J] \leq [D, E]$ and $[I, J] \leq [F, G]$

$\implies I \leq D$ and $J \leq E$ and $I \leq F$ and $J \leq G$ (Theorem 70)

$\implies I \leq D \wedge F$ and $J \leq E \wedge G$ (Induction Hypothesis)

$\implies [I, J] \leq [D \wedge F, E \wedge G]$ (Theorem 70)

$\implies [I, J] \leq [D, E] \wedge [F, G]$ (Definition 111).

$\square$

Definition 113. $\vee$ is defined by recursion. In C_0, it is as follows:

$\perp_0 \vee \perp_0 := \perp_0$

$\perp_0 \vee \top_0 := \top_0$

$\top_0 \vee \perp_0 := \top_0$

$\top_0 \vee \top_0 := \top_0$

If A and B are objects of C_n where n is a positive integer.

$A \vee B := [Dom_n(A) \vee Dom_n(B), Cod_n(A) \vee Cod_n(B)]$.

Comments 114. $A \wedge B$ is the glb of A and B and $A \vee B$ is the lub of A and B.

$\wedge$ is also the category-theoretic product (along with the (projection) arrows that 'say' (Definition 66) that $A \wedge B \leq A$ and $A \wedge B \leq B$). $\vee$ is the category theoretic sum (along with the arrows that 'say' (Definition 66) that $A \leq A \vee B$ and $B \leq A \vee B$).

$A \wedge B$ will also turn out to be the game-theoretic meet (or product) of the SVGs A and B, the game that passes the bill iff A and B pass the bill while $A \vee B$ is the game-theoretic sum.

Comments 115. The next result states that $\wedge$ and $\vee$ are dual to each other.

Theorem 116. $(A \vee B)^* = A^* \wedge B^*$

Proof. The proof is by induction on n. In C_0 it is proved by cases.

First $A = \bot_0$ and $B = \bot_0$

$(\bot_0 \vee \bot_0)^*$

$= \bot_0^*$ (Definition 113)

$= \top_0$ (Theorem 72)

$= (\top_0 \wedge \top_0)$ (Definition 111)

$= \bot_0^* \wedge \bot_0^*$ (Theorem 72)

Second $A = \bot_0$ and $B = \top_0$

$(\bot_0 \vee \top_0)^*$

$= \top_0^*$ (Definition 113)

$= \bot_0$ (Theorem 72)

$= (\top_0 \wedge \bot_0)$ (Definition 111)

$= \bot_0^* \wedge \top_0^*$ (Theorem 72)

Third $A = \top_0$ and $B = \bot_0$

$(\top_0 \vee \bot_0)^*$

$= \top_0^*$ (Definition 113)

$= \bot_0$ (Theorem 72)

$= (\bot_0 \wedge \top_0)$ (Definition 111)

$= \top_0^* \wedge \bot_0^*$ (Theorem 72)

By Theorem 118, $A \leq A \vee B$

Fourth $A = \top_0$ and $B = \top_0$

$(\top_0 \vee \top_0)^*$

$= \top_0^*$ (Definition 113)

$= \perp_0$ (Theorem 72)

$= (\perp_0 \wedge \perp_0)$ (Definition 111)

$= \top_0^* \wedge \top_0^*$ (Theorem 72)

Let us assume that the result is true for $k = n$.

Let $A = [D, E]$ and $B = [F, G]$

$(A \vee B)^*$

$= ([D, E] \vee [F, G])^*$

$= ([D \vee F, E \vee G])^*$ (Definition 113)

$= ([(E \vee G)^*, (D \vee F)^*])$ (Theorem 72)

$= ([(E^* \wedge G^*), (D^* \wedge F^*)])$ (Induction Hypothesis)

$= ([(E^*, D^*] \wedge [G^*, F^*)])$ (Definition 111)

$= ([D, E]^* \wedge [F, G]^*)$ (Theorem 72)

$= (A^* \wedge B^*)$

□

Theorem 117. $(A \wedge B)^* = A^* \vee B^*$

Proof. This is the dual of Theorem 116

□

Theorem 118. *$A \vee B$ is the least upper bound of A and B.*

Proof. This is the dual of Theorem 112.

□

Comments 119. Theorem 71 tells us that any diagram of arrows in C_n must commute. For example consider this diagram:

$$\begin{tikzcd}
D \arrow[r, "g"'] & E \\
B \arrow[u, "h"'] \arrow[r, "f"'] & C \arrow[u, "i"']
\end{tikzcd}$$

The composition of g and h must be an arrow from B to E. So must the composite of i and f. Since there cannot be more than one arrow from B to E, the two composites are equal and the diagram commutes. It is clear that this reasoning generalises to all diagrams.

$$\begin{tikzcd}
D \arrow[r, "g"'] & E \\
B \arrow[u, "h"'] \arrow[r, "f"'] \arrow[ur, "gh = if"'] & C \arrow[u, "i"']
\end{tikzcd}$$

Theorem 120. *C_n has all finite products.*

If A and B are objects of C_n then $A \wedge B$ is the product of A and B. The arrows are the ones that result from the fact that $A \geq A \wedge B$ and $B \geq A \wedge B$ (Definition 66).

Proof. It is a standard result of category theory that, in an order category, the glb is also the product but I have included the proof for completeness.

The proof is by induction on n.

First, let us start with $n = 0$.

Definition 111 tells us that $\perp \wedge \perp = \perp$ so we need to show that this is a product diagram

$$
\begin{tikzcd}
\perp & \perp \arrow[l, "1_\perp"'] \arrow[r, "1_\perp"] & \perp
\end{tikzcd}
$$

If we have two arrows to $\perp$, the domain must be $\perp$, the arrows must be $1_\perp$ and the unique u is $1_\perp$.

$$
\begin{tikzcd}
 & \perp \arrow[dl, "1_\perp"'] \arrow[d, "1_\perp"'] \arrow[dr, "1_\perp"] & \\
\perp & \perp \arrow[l, "1_\perp"] \arrow[r, "1_\perp"'] & \perp
\end{tikzcd}
$$

The following diagrams cover all of the other possible cases.

$$
\begin{tikzcd}
 & \perp \arrow[dl, "1_\perp"'] \arrow[d, "1_\perp"'] \arrow[dr, "{[\perp, \top]}"] & \\
\perp & \perp \arrow[l, "1_\perp"] \arrow[r, "{[\perp, \top]}"'] & \top
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \bot \arrow[dl, "{[\bot,\top]}"'] \arrow[d, "{[\bot,\top]}"'] \arrow[dr, "{[\bot,\top]}"] & \\
\top & \top \arrow[l, "1_\top"] \arrow[r, "1_\top"'] & \top
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \top \arrow[dl, "1_\top"'] \arrow[d, "1_\top"'] \arrow[dr, "1_\top"] & \\
\top & \top \arrow[l, "1_\top"] \arrow[r, "1_\top"'] & \top
\end{tikzcd}
$$

Of course these diagrams just express the not-too-surprising fact that if $A \leq B$ and $A \leq C$ then $A \leq$ the greatest lower bound of B and C. Of course this is just the definition of greatest lower bound.

Let us assume that, for all A, B, this is a product diagram for A and B. C_k.

$$
\begin{tikzcd}
A & A \wedge B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}
$$

We need to show that we have a product diagram

$$
\begin{tikzcd}
{[C,D]} & {[C,D] \wedge [E,F]} \arrow[l, "p_1"'] \arrow[r, "p_2"] & {[E,F]}
\end{tikzcd}
$$

for all objects $[C, D]$ and $[E, F]$ of C_{k+1}.

The induction hypothesis tells us that we have two product diagrams:

$$
\begin{tikzcd}
C & C \wedge E \arrow[l, "q_1"'] \arrow[r, "q_2"] & E
\end{tikzcd}
$$

$$
\begin{tikzcd}
D & D \wedge F \arrow[l, "r_1"'] \arrow[r, "r_2"] & F
\end{tikzcd}
$$

The definition of $\wedge$ (Definition 111) tells us that $[C, D] \wedge [E, F] = [C \wedge E, D \wedge F]$

Theorem 70 tells us that an arrow from $C \wedge E$ to C and an arrow from $D \wedge F$ to D gives us an arrow from $[C \wedge E, D \wedge F] = [C, D] \wedge [E, F]$ to $[C, D]$.

In a similar way, we have an arrow from $[C \wedge E, D \wedge F]$ to $[E, F]$. So we have this diagram.

$$
\begin{tikzcd}
{[C,D]} & {[C,D] \wedge [E,F]} \arrow[l, "{[q_1, r_1]}"'] \arrow[r, "{[q_2, r_2]}"] & {[E,F]}
\end{tikzcd}
$$

Now let us say that we have an object $[S, T]$ and arrows $[f, g] : [S, T] \to [C, D]$ and $[i, j] : [S, T] \to [E, F]$.

$$\begin{tikzcd}
 & {[S,T]} \arrow[dl, "{[f,g]}"'] \arrow[d, "{[u,v]}"] \arrow[dr, "{[i,j]}"] & \\
{[C,D]} & {[C,D] \wedge [E,F]} \arrow[l, "{[q_1,r_1]}"] \arrow[r, "{[q_2,r_2]}"'] & {[E,F]}
\end{tikzcd}$$

Theorem 70 then gives us two diagrams.

$$\begin{tikzcd}
 & S \arrow[dl, "f"'] \arrow[d, "u"] \arrow[dr, "i"] & \\
C & C \wedge E \arrow[l, "q_1"] \arrow[r, "q_2"'] & E
\end{tikzcd}$$

Which, with the induction hypothesis, gives us a unique $u : S \to C \wedge E$

and

$$\begin{tikzcd}
 & T \arrow[dl, "g"'] \arrow[d, "v"] \arrow[dr, "j"] & \\
D & D \wedge F \arrow[l, "r_1"] \arrow[r, "r_2"'] & F
\end{tikzcd}$$

Which, again with the induction hypothesis, gives us a unique $v : T \to D \wedge F$

Putting these together, we have a unique $[u, v] : [S, T] \to [C \wedge E, D \wedge F]$

that makes the original diagram commute.

$\square$

Theorem 121. *C_n has all equalizers.*

Proof. For every parallel pair of arrows.

$$\begin{tikzcd} A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & B \end{tikzcd}$$

We need $e : E \to A$ such that $fe = ge$ and given $t : T \to A$ such that $ft = gt$ there is a unique u that makes the following diagram commute. [4, Page 30]

$$\begin{tikzcd} T \arrow[d, "u"'] \arrow[dr, "t"] & & \\ E \arrow[r, "e"'] & A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & B \end{tikzcd}$$

In the case of our category, we know that if f and g have the same range and domain then $f = g$. In this case, it can be seen from the diagram that $1_A : A \to A$ can play the role of $e : E \to A$ (for each t, $u = t$).

$\square$

Theorem 122. *C_n has all finite limits.*

Proof. Every category that has a terminator, all products and all equalisers will have all limits ([4, Theorem 4.11]). So Theorem 120 and Theorem 121 give us the result.

There is also another way of looking at this. Consider a diagram D with vertices $A_i : (1 \leq i \leq n)$ and edges $f_{i,j} : A_i \to A_j$.

A cone ([4, P48]) consists of an object C such that $(\forall i), C \leq A_i$ and the arrows that say that $(\forall i), C \leq A_i$ (call them g_i). The commuting conditions ($f_{i,j} g_i = g_j$) are satisfied automatically because any diagram within C_n commutes (Comment 119).

A limit is a cone with an arrow to it from every other cone (again the commuting conditions are all satisfied automatically). So it is defined by the largest object that it less than or equal to all the A_i or just $A_1 \wedge A_2 \wedge \cdots \wedge A_n$

□

Comments 123. As a special case of the Theorem above, the pullback of a corner of arrows.

$$\begin{tikzcd} A \arrow[r, "l_1"'] & C \\ & B \arrow[u, "l_2"] \end{tikzcd}$$

Is the product of A and B.

Theorem 124. *C_n has all finite coproducts,*

The coproduct of A and B is just $A \vee B$ and the arrows that say that $A \leq A \vee B$ and $B \leq A \vee B$.

Proof. This is the dual of Theorem 120.

□

Theorem 125. *C_n as all coequalizers*

For any parallel pair of arrows f and g.

$$\begin{tikzcd} A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & B \end{tikzcd}$$

$f = g$ and the coequalizer of f and g is 1_B.

Proof. The is the dual of Theorem 121.

□

Theorem 126. *Every diagram has a colimit. The object of this colimit is the coproduct of the objects of the diagram.*

Proof. This is the dual of Theorem 122.

□

Theorem 127. *Every arrow in all of the C_n is epic and monic. The non-identity arrows are not iso.*

Proof. The fact that every arrow is epic and monic springs from the fact that there is at most one arrow between any two objects. As does the fact that they are not iso (to be iso we would need an arrow going back the other way)

□

Theorem 128. *Every object in C_n is a subterminal object (and a subinitial object) [4, Exercise 2.12]*

Proof. This is a direct consequence of the fact that there is not more then one arrow between any two objects (Theorem 71).

□

Comments 129. So to collect these comments. The C_n have all limits (including terminal object, products, pullbacks and equalizers) and all colimits (initial objects, coproducts, pushouts and coequalizers). Also every non-identity arrow is epic and monic but not iso and every object is subterminal and initial.

All of this really springs from the fact that the C_n are partially ordered sets and so there is not more than one arrow between any two objects. The presence of terminal and initial and objects tells us that the partially ordered set has maximal and minimal objects and the existence of products and coproducts corresponds to the fact that there are greatest lower bounds and least upper bounds (respectively) for every set of objects.

Theorem 130. *$A \wedge B = \perp_n \iff (A = \perp_n$ or $B = \perp_n)$.*

Proof. $A = \perp_n$

$\implies A \wedge B = \perp_n$ (Theorem 104 and Theorem 112)

Of course, the implication for $B = \perp_n$ is identical.

The proof of the implication in the other direction is by induction.

In C_0 we have. $\top \wedge \top = \top$; $\top \wedge \perp = \perp$; $\perp \wedge \top = \perp$ and $\perp \wedge \perp = \perp$.

Let us assume that we have the theorem in C_k.

$A \wedge B = \perp_{k+1}$

$\implies Cod_{k+1}(A) \wedge Cod_{k+1}(B) = Cod_{k+1}(\perp_{k+1})$ (Definition 111)

$\implies Cod_{k+1}(A) \wedge Cod_{k+1}(B) = \perp_k$ (Definition 85)

$\implies Cod_{k+1}(A) = \perp_k$ (By the induction hypothesis and without loss of generality)

$\implies Dom_{k+1}(A) = \perp_k$ (Theorem 104, Theorem 68)

$\implies A = \perp_{k+1}$ (Definition 85)

$\square$

Theorem 131. *$A \vee B = \top_n \iff (A = \top_n$ or $B = \top_n)$.*

Proof. This is the dual of Theorem 130.

$\square$

Theorem 132. *If A, B and D are objects of C_n then $(A \vee B) \wedge D = (A \wedge D) \vee (B \wedge D)$.*

Proof. First, these are all the possibilities in C_0:

$(\bot_0 \vee \bot_0) \wedge \bot_0 = (\bot_0 \wedge \bot_0) \vee (\bot_0 \wedge \bot_0) = \bot_0$

$(\bot_0 \vee \bot_0) \wedge \top_0 = (\bot_0 \wedge \top_0) \vee (\bot_0 \wedge \top_0) = \bot_0$

$(\bot_0 \vee \top_0) \wedge \bot_0 = (\bot_0 \wedge \bot_0) \vee (\top_0 \wedge \bot_0) = \bot_0$

$(\bot_0 \vee \top_0) \wedge \top_0 = (\bot_0 \wedge \top_0) \vee (\top_0 \wedge \top_0) = \top_0$

$(\top_0 \vee \bot_0) \wedge \bot_0 = (\top_0 \wedge \bot_0) \vee (\bot_0 \wedge \bot_0) = \bot_0$

$(\top_0 \vee \bot_0) \wedge \top_0 = (\top_0 \wedge \top_0) \vee (\bot_0 \wedge \top_0) = \top_0$

$(\top_0 \vee \top_0) \wedge \bot_0 = (\top_0 \wedge \bot_0) \vee (\top_0 \wedge \bot_0) = \bot_0$

$(\top_0 \vee \top_0) \wedge \top_0 = (\top_0 \wedge \top_0) \vee (\top_0 \wedge \top_0) = \top_0$

Let us assume that we have the result for $n = k$. Let $A = [E, F]$, $B = [G, H]$ and $D = [I, J]$ be objects of C_{k+1}.

$(A \vee B) \wedge C$

$= ([E, F] \vee [G, H]) \wedge [I, J]$

$= ([E \vee G, F \vee H]) \wedge [I, J]$ Definition 113

$= [(E \vee G) \wedge I, (F \vee H) \wedge J]$ Definition 111

$= [(E \wedge I) \vee (G \wedge I), (F \wedge J) \vee (H \wedge J)]$ Induction Hypothesis

$= [E \wedge I, F \wedge J] \vee [G \wedge I, H \wedge J]$ Definition 113

$= ([E, F] \wedge [I, J]) \vee ([G, H] \wedge [I, J])$ Definition 111

$= (A \wedge D) \vee (B \wedge D)$

This completes the induction and the proof.

□

Theorem 133. *If A, B and D are objects of C_n then $(A \wedge B) \vee D = (A \vee D) \wedge (B \vee D)$.*

Proof. This is the dual of Theorem 132

□

Theorem 134. *$A \leq B$ iff $B = A \vee B$.*

Proof. We could just say that this follows from the fact that $A \vee B$ is the least upper bound but here is the proof.

First from right to left. $B = A \vee B$ tells us that $A \leq B$ because $A \leq A \vee B$ (Theorem 118). The other direction is new.

First I will check it is true in C_0.

$\bot_0 \leq \bot_0$ and $\bot_0 = \bot_0 \vee \bot_0$

$\bot_0 \leq \top_0$ and $\top_0 = \bot_0 \vee \top_0$

$\top_0 \leq \top_0$ and $\top_0 = \top_0 \vee \top_0$

$\neg(\top_0 \leq \bot_0)$ and $\neg(\bot_0 = \top_0 \vee \bot_0)$

Let us assume that the result is true for $n = k$.

Let $A = [D, E]$ and $B = [F, G]$ be objects of C_{k+1}.

$A \leq B$

$\iff [D, E] \leq [F, G]$

$\iff D \leq F$ and $E \leq G$ Theorem 70

$\iff D = D \vee F$ and $E = E \vee G$ Induction Hypothesis

$\iff [D, E] = [D \vee F, E \vee G]$

$\iff [D, E] = [D, E] \vee [F, G]$ Definition 113

$\iff A = A \vee B$

And so we have completed the induction and the proof.

□

Theorem 135. *$A \leq B$ iff $A = A \wedge B$.*

Proof. This is the dual of Theorem 134

□

Theorem 136. *If A is any object of C_n then $A \wedge A = A$ and $A \vee A = A$.*

Proof. This is obvious when we think of $\wedge$ and $\vee$ as glb and lub respectively. The results are also dual to each other.

There is an identity arrow from A to itself.

Definition 66 tells us that $A \leq A$.

Theorem 135 then tells us that $A = A \wedge A$

Theorem 134 tells us that $A = A \vee A$

□

Theorem 137. *If A is an object of C_n then $A = Dum_{n-1}(Dom_n(A)) \vee (Dict_n \wedge Dum_{n-1}(Cod_n(A)))$*

Proof. $Dum_{n-1}(Dom_n(A)) \vee (Dict_n \wedge Dum_{n-1}(Cod_n(A)))$

$= Dum_{n-1}(Dom_n(A)) \vee ([\perp_{n-1}, \top_{n-1}] \wedge Dum_{n-1}(Cod_n(A)))$ (Definition 110)

$= [Dom(A), Dom(A)] \vee ([\perp_{n-1}, \top_{n-1}] \wedge [Cod(A), Cod(A)]) =$ (Definition 83)

$= [Dom(A), Dom(A)] \vee ([\perp_{n-1} \wedge Cod(A), \top_{n-1} \wedge Cod(A)]$ (Definition 111)

$= [Dom(A), Dom(A)] \vee ([\perp_{n-1}, Cod(A)]$ (Theorem 105, Theorem 104 and Theorem 112)

$= [Dom(A) \vee \perp_{n-1}, Dom(A) \vee Cod(A)]$ (Definition 111)

$= [Dom(A), Cod(A)]$ (Theorem 104, Theorem 118, Theorem 68)

$= A$ (Definition 92 and 94)

□

Theorem 138. *$Dict_n$, an object of C_n, is self-dual.*

Proof. $Dict_n^*$

$= [Cod_n(Dict_n)^*, Dom_n(Dict_n)^*]$ (Theorem 72)

$= [\top_{n-1}^*, \perp_{n-1}^*]$ (Definition 110)

$= [\perp_{n-1}, \top_{n-1}]$ (Theorem 87)

$= Dict_n$ (Definition 110)

□

4.3 Bipartitions in the Lattice Category of Simple Voting Games

Definition 139. A *principal object* is an object A, of C_n, such that, if $A = D \vee E$ then $A = D$ or $A = E$. Equally, this could be called an irreducible object.

Definition 140. A *prime object* is an object A, of C_n, such that, if $A = D \wedge E$ then $A = D$ or $A = E$.

Theorem 141. *If A is principal then A^* is prime. If A is prime then A^* is principal.*

Proof. Let us assume that A is principal. Assume $A^* = D \wedge E$. $A = D^* \vee E^*$ (By Theorem 117 and Theorem 75). So $A = D^*$ or $A = E^*$. In the first case $A^* = D$ and in the second $A^* = E$. (Using Theorem 75)

Let us assume that A is prime. If $A^* = D \vee E$ then $A = D^* \wedge E^*$ (By Theorem 116 and Theorem 75). So either $A = D^*$ or $A = E^*$. In the first case $A^* = D$. In the second $A^* = E$. (Using Theorem 75)

□

Theorem 142. $\top_n$ *is a principal object and a prime object.*

Proof. By Definition 139, we are trying to prove that:

If E and F are objects of C_n then $E \vee F = \top_n$ implies that $E = \top_n$ or $F = \top_n$

The proof that $\top_n$ is principal is by induction on n. In C_0

$\top_0 \vee \top_0 = \top_0$

$\top_0 \vee \bot_0 = \top_0$

$\bot_0 \vee \bot_0 = \bot_0$

Let us assume that each is true for $n = k$.

Let A and B be objects of C_{k+1} with $A \vee B = \top_{k+1}$. Definition 113 and Definition 85 tell us that $Dom_{k+1}(A) \vee Dom_{k+1}(B) = \top_k$ and $Cod_{k+1}(A) \vee Cod_{k+1}(B) = \top_k$. The induction hypothesis tells us that $Dom_{k+1}(A)$ or $Dom_{k+1}(B)$ are equal to $\top_k$. Let us assume WLOG that it is $Dom_{k+1}(A)$. Theorem 68 tells us that $Dom_{k+1}(A) \leq Cod_{k+1}(A)$ and Theorem 105 tells us that $Cod_{k+1}(A) = Dom_{k+1}(A) = \top_k$. Finally (Using Definition 92, Definition 94 and Definition 85), $A = \top_k$

To show that $\top_n$ is prime, assume that $A \wedge B = \top_n$. $\wedge$ is the greatest lower bound (Theorem 112) and so it must be the case that A and B are both $\top_n$.

□

Theorem 143. $\bot_n$ *is a principal object and a prime object.*

Proof. This is dual to Theorem 142.

□

Theorem 144. $Dict_n$ *is a principal object and a prime object.*

Proof. Let us say that $A \vee B = Dict_n$.

The definition of $Dict_n$ (Definition 110) and Defintion 113 tell us that $Dom_{n-1}(A) \vee Dom_{n-1}(B) = \perp_{n-1}$ and $Cod_{n-1}(A) \vee Cod_{n-1}(B) = \top_{n-1}$. $\vee$ is a least upper bound (Theorem 118) and $\perp_{n-1}$ is minimal in C_{n-1} (Theorem 104). This tells us that $Dom_{n-1}(A) = Dom_{n-1}(B) = \perp_{n-1}$. $\top_{n-1}$ is principal (Theorem 142) and so $Cod_{n-1}(A)$ or $Cod_{n-1}(B)$ are equal to $\top_{n-1}$ hence A or B is equal to $Dict_n$.

□

Definition 145. A *bipartition* is an object A of C_n that is principal and not equal to $\perp_n$.

Comments 146. We have used the word 'bipartition' earlier in the document (Definition 47) to refer to, what looked like a very different sort of mathematical object. We will see that the two meanings match.

Definition 147. An *inverted bipartition* is an object A that is prime and not equal to $\top_n$.

Theorem 148. *$Dict_n$ is a bipartition and an inverted bipartition.*

Proof. $Dict_n$ is a principal object (Theorem 144). $Dict_n \neq \perp_n$ (Definition 110 and Definition 85). So $Dict_n$ is a bipartition (Definition 145).

$Dict_n$ is a prime object (Theorem 144). $Dict_n \neq \perp_n$ (Definition 110 and Definition 85). So $Dict_n$ is an inverted bipartition (Definition 147). Since $Dict_n$ is self-dual, we could just have announced that this is the dual result.

□

Comments 149. We will see later that C_n has n objects that, like $Dict_n$, are principal and prime and not equal to $\bot_n$ or $\top_n$. They play the role of the voters.

Theorem 150. *Vet_n maps principal elements in C_n to principal elements in C_{n+1}.*

Proof. Let A be a principal object of C_n.

$Vet_n(A) = D \vee E$

$\implies [\bot_n, A] = D \vee E$ (Using Definition 88.)

$\implies$ $Dom_n(D \vee E) = \bot_n$ and $Cod_n(D \vee E) = A$ (Definition 92 and Definition 94)

$\implies Dom_n(D) \vee Dom_n(E) = \bot_n$ and $Cod_n(D) \vee Cod_n(E) = A$. (Using Definition 113)

Theorem 104 and Theorem 118 applied to this first part of the conjunction tell us that.

$Dom_n(D) = \bot_{n-1}$ and $Dom_n(E) = \bot_{n-1}$.

A is principal and so $Cod_n(D) = A$ or $Cod_n(E) = A$.

Hence either D or E is equal to $[\bot_{n-1}, A] = Vet_n(A)$ and $Vet_n(A)$ is principal.

□

Vet_n does not, in general, map prime elements to prime elements.

$Dict_1 = [\bot_0\top_0]$ is prime but $Vet_1([\bot_0\top_0]) = [\bot_0\bot_0\bot_0\top_0] = [\bot_0\top_0\bot_0\top_0] \wedge [\bot_0\bot_0\top_0\top_0]$ and so it is not prime.

Theorem 151. *Vet_n maps bipartitions in C_n to bipartitions in C_{n+1}.*

Proof. If A is a bipartition then it is a principal element (Definition 145).

$Vet_n(A)$ is a principal object (Theorem 150).

To show that $Vet_n(A)$ is a bipartition, I need to show that it is not $\perp_n$ (Definition 145).

Let us say that $Vet_n(A) = \perp_n$. That would require A to be $\perp_{n-1}$ (Definition 88 and Definition 85). This is not possible as $\perp_{n-1}$ is not a bipartition (Definition 145).

$\square$

Comments 152. Vet_n does not map inverted bipartitions to inverted bipartitions. The counterexample that was used for prime elements also works for inverted bipartitions (every inverted bipartition is prime by Definition 147).

Vet_n does not respect duals. That would require $Vet_n(A^*) = Vet_n(A)^*$.

In fact $Vet_n(A^*) = Pas_n(A)^*$ (Theorem 97)

Theorem 153. *Pas_n maps prime elements in C_n to prime elements in C_{n+1}.*

Proof. This is the dual of Theorem 150.

$\square$

Pas_n does not, in general, map principal elements to principal elements.

$Dict_1 = [\perp_0 \top_0]$ is principal but $Pas_1([\perp_0 \top_0]) = [\perp_0 \top_0 \top_0 \top_0] = [\perp_0 \top_0 \perp_0 \top_0] \vee [\perp_0 \perp_0 \top_0 \top_0]$ and so it is not.

Theorem 154. *Pas_n maps inverted bipartitions in C_n to inverted bipartitions in C_{n+1}.*

Proof. This is the dual of Theorem 151

$\square$

Comments 155. Pas_n does not map bipartitions to bipartitions. The counterexample that was used for principal elements also works for bipartitions (every bipartition is principal by Definition 145).

Pas_n does not respect duals. That would require $Pas_n(A^*) = Pas_n(A)^*$.

In fact $Vet_n(A^*) = Pas_n(A)^*$ (Theorem 97)

Theorem 156. *Dum_n maps principal elements in C_n to principal elements in C_{n+1}.*

Proof. The proof is by induction. First, let $n = 0$. $Dum_n(\bot_0) = \bot_1$ and $Dum_n(\top_0) = \top_1$. These are both principal (Theorem 142 and Theorem 143)

Let A be a principal object of C_n.

$Dum_n(A) = D \vee E$

$\implies [A, A] = D \vee E$ (Definition 83)

$\implies Dom_n(D \vee E) = A$ and $Cod_n(D \vee E) = A$ (Definition 92 and Definition 94)

$\implies Dom_n(D) \vee Dom_n(E) = A$ and $Cod_n(D) \vee Cod_n(E) = A$ (Definition 113)

$\implies (Dom_n(D) = A$ or $Dom_n(E) = A)$ and $(Cod_n(D) = A$ or $Cod_n(E) = A)$ (By assumption, A is principal)

Let us say, WLOG, $Dom_n(D) = A$. If $Cod_n(D) = A$ then we are done.

Otherwise $Cod_n(E) = A$.

In this case, $Cod_n(D) \vee Cod_n(E) = A$ tells us that $Cod_n(D) \vee A = A$. This and Theorem 134 give us $Cod_n(D) \leq A$.

Theorem 68, Definition 92 and Definition 94 tell us that $A = Dom_n(D) \leq Cod_n(D)$.

These two inequalities and Theorem 81 tell us that $Cod_n(D) = A$ and (by Definition 139) $Dum_n(A)$ is principal.

□

Theorem 157. *Dum_n maps prime elements in C_n to prime elements in C_{n+1}.*

Proof. This is dual to Theorem 156.

□

Theorem 158. *Dum_n maps bipartitions in C_n to bipartitions in C_{n+1}.*

Proof. Let A be a bipartition in C_n. Definition 145 tells us that,

A is a principal object of C_n. Theorem 156 tells us that,

$Dum_n(A)$ is a principal object of C_{n+1}. $Dum_n(A)$ is a bipartition as long as it is not equal to $\perp_{n+1}$ (Definition 145).

$Dum_n(A) = \perp_{n+1}$

$\implies [A, A] = [\perp_n, \perp_n]$ (Using Definition 83 and Definition 85).

$\implies A = \perp_n$. This is not possible as A is a bipartition (Definition 145).

□

Theorem 159. *Dum_n maps inverted bipartitions in C_n to inverted bipartitions in C_{n+1}.*

Proof. This is just the dual of Theorem 158

□

Theorem 160. *Dom_n maps principal elements in C_n to principal elements in C_{n-1}.*

Proof. Let A be a principal object of C_n.

Let $Dom_n(A) = D \vee E$.

$A = [Dom_n(A), Cod_n(A)]$ Using Definition 92 and Definition 94.

$A = [D \vee E, Cod_n(A) \vee Cod_n(A)]$ Theorem 136

$A = [D, Cod_n(A)] \vee [E, Cod_n(A)]$. Using Definition 113.

A is principal and so $A = [Dom_n(A), Cod_n(A)] = [D, Cod_n(A)]$ or $[E, Cod_n(A)]$. Hence $Dom_n(A) = D$ or $Dom_n(A) = E$ and $Dom_n(A)$ is principal.

□

Theorem 161. *Dom_n maps prime elements in C_n to prime elements in C_{n-1}.*

Proof. Let A be a prime object of C_n.

Let $Dom_n(A) = D \wedge E$.

$A = [Dom_n(A), Cod_n(A)]$ (Definition 92 and Definition 94).

$A = [D \wedge E, Cod_n(A) \wedge Cod_n(A)]$ (Theorem 136)

$A = [D, Cod_n(A)] \wedge [E, Cod_n(A)]$ (Definition 111)

A is prime and so $A = [Dom_n(A), Cod_n(A)] = [D, Cod_n(A)]$ or $[E, Cod_n(A)]$. Hence $Dom_n(A) = D$ or $Dom_n(A) = E$ and $Dom_n(A)$ is prime.

□

Dom_n does not map bipartitions to bipartitions. Let A be any bipartition in C_n. $Vet_n(A)$ is a bipartition in C_{n+1} (By Theorem 151). $Dom_{n+1}(Vet_n(A)) = \perp_n$ (By Theorem 88 and Definition 92). $\perp_n$ is not a bipartition (Definition 145).

Theorem 162. *Dom_n maps inverted bipartitions in C_n to inverted bipartitions in C_{n-1}.*

Proof. If A is an inverted bipartition then it is prime (Definition 147). Theorem 161 tells us that $Dom_n(A)$ is prime. To show that it is an inverted bipartition, I need to show that $Dom_n(A) \neq \top_{n-1}$. Let us say that $Dom_n(A) = \top_{n-1}$. Theorem 68 tells us that $Cod_n(A) \geq Dom_n(A)$. $Dom_n(A) = \top_{n-1}$ and

$\top_{n-1}$ is maximal in C_n (Theorem 105) and so $Cod_n(A) = \top_{n-1}$. In this case, $A = \top_n$ (Definition 85, Definition 92 and Definition 94). If this were the case then A wouldn't be an inverted bipartition (Definition 145).

□

Dom_n does not respect the operation of taking the dual. This would require $Dom_n(A^*) = Dom_n(A)^*$. In general, this is not true. In fact, of course:

$Dom_n(A^*) = Cod_n(A)^*$ (Theorem 99)

Theorem 163. *Cod_n maps principal elements in C_n to principal elements in C_{n-1}.*

Proof. This is the dual of Theorem 161

□

Theorem 164. *Cod_n maps prime elements in C_n to prime elements in C_{n-1}.*

Proof. This is the dual of Theorem 160

□

Theorem 165. *Cod_n maps bipartitions in C_n to bipartitions in C_{n-1}.*

Proof. This is the dual of Theorem 162

□

Comments 166. Cod_n does not map inverted bipartitions to inverted bipartitions. Let A be any inverted bipartition in C_n. $Pas_n(A)$ is an inverted bipartition in C_{n+1} (By Theorem 154). $Cod_{n+1}(Pas_n(A)) = \top_n$ (By Theorem 90 and Definition 94). $\top_n$ is not an inverted bipartition (Definition 147).

Cod_n does not respect the operation of taking the dual. This would require $Cod_n(A^*) = Cod_n(A)^*$. In fact

$Cod_n(A^*) = Dom_n(A)^*$ (Theorem 100)

The results on what Vet_n, Pas_n, Dum_n, Dom_n and Cod_n preserve can be summarised as follows.

Respecting	Vet_n	Pas_n	Dum_n	Dom_n	Cod_n
$\vee$ and $\wedge$	Y	Y	Y	Y	Y
$\top_n$	N	Y	Y	Y	Y
$\perp_n$	Y	N	Y	Y	Y
principal	Y	N	Y	Y	Y
prime	N	Y	Y	Y	Y
bipartition	Y	N	Y	N	Y
inv bip	N	Y	Y	Y	N
dual	N	N	Y	N	N

Comments 167. The next four theorems will be used a lot in the rest of the paper.

Theorem 168. *If A is a principal object of C_n then $A = Vet_n(B)$ or $A = Dum_n(B)$ with B principal.*

Proof. Let us assume that A is principal in C_n.

By Theorem 137, $A = Dum_{n+1}(Dom_n(A)) \vee (Dict_n \wedge Dum_{n+1}(Cod_n(A)))$.

Since A is principal, either $A = Dum_{n+1}(Dom_n(A))$ or $A = Dict_n \wedge Dum_{n+1}(Cod_n(A)))) = [\perp_{n-1}, \top_{n-1}] \wedge Dum_{n+1}(Cod_n(A)))$.

Definition 111 tells us that $[\perp_{n-1}, \top_{n-1}] \wedge Dum_{n+1}(Cod_n(A))] = [\perp_{n-1} \wedge Cod_n(A), \top_{n-1} \wedge Cod_n(A)]$.

$\perp_{n-1}$ is minimal (Theorem 104), $\top_{n-1}$ is maximal (Theorem 105) and $\wedge$ is the glb (Theorem 112).

This tells us that $[\perp_{n-1} \wedge Cod_n(A), \top_{n-1} \wedge Cod_n(A)] = [\perp_{n-1}, Cod_n(A)] = Vet_{n+1}(Cod_n(A))$ (Using Definition 88)

So I have shown that $A = Dum_{n+1}(Dom_n(A))$ or $A = Vet_{n+1}(Cod_n(A))$.

I need to show that $Dom_n(A)$ is principal (assuming that A is). This is just Theorem 160. I also need to show that $Cod_n(A)$ is principal. This is Theorem 163.

$\square$

Theorem 169. *If A is a prime object in C_n then $A = Dum_n(B)$ or $A = Pas_n(B)$ with B prime.*

Proof. This is the dual of Theorem 168.

$\square$

Comments 170. So every principal element is of the form $Vet_n(D)$ or $Dum_n(D)$ with D principal (Theorem 168). Every element of the form $Vet_n(D)$ or $Dum_n(D)$, with D principal, is principal (Theorem 150 and Theorem 156).

And every prime element is of the form $Pas_n(D)$ or $Dum_n(D)$ with D prime (Theorem 169). Every element of the form $Pas_n(D)$ or $Dum_n(D)$, with D prime, is prime (Theorem 153 and Theorem 157).

Theorem 171. *The bipartitions of C_{n+1} are exactly, the objects $Vet_n(A)$ and $Dum_n(A)$ where A is a bipartition of C_n. The only bipartition in C_0 is $\top_0$.*

Proof. A is principal iff it is of the form $Vet_n(B)$ or $Dum_n(B)$ where B is principal. (Theorem 168, Theorem 150 and Theorem 156). In both cases, this equals $\perp_n$ iff $A = \perp_{n-1}$ (Definition 88, Definition 90 and Definition 85). And so $Vet_n(B)$ and $Dum_n(B)$ are bipartitions iff $A \neq \perp_{n-1}$. i.e. iff B is a bipartition.

□

Comments 172. So we will allow the unanimity games to play the role of the bipartitions. The 'meaning' is given to them by recursion. If A is a bipartition of C_n and $A = Dum_n(B)$ then the n^{th} voter votes 'no' in A. If $A = Vet_n(B)$ then the n^{th} voter votes 'yes' in A. The same analysis in the next smallest category will tell us whether v_{n-1} is in B.

Theorem 173. *The inverted bipartitions of C_{n+1} are exactly, the objects $Pas_n(A)$ and $Dum_n(A)$ where A is an inverted bipartition of C_n. The only inverted bipartition in C_0 is $\perp_0$.*

Proof. This is the dual of Theorem 171.

□

Comments 174. We will also allow the duals of the unanimity games to provide a second copy of the bipartitions - referring to them as 'inverted bipartitions' because the structure is the other way up. Again, the 'meaning' is given to them by recursion. If A is a bipartition of C_n and $A = Dum_n(B)$ then the n^{th} voter votes 'no' in A. If $A = Pas_n(B)$ then the n^{th} voter votes 'yes' in A. The same analysis in the next smallest category will tell us whether v_{n-1} is in B.

Theorem 175. *Taking the dual bijectively maps bipartitions to inverted bipartitions and vice versa.*

Proof. Let B be a bipartition. Definition 145 tell us that it is principal and not equal to $\bot_{n-1}$. Theorem 141 tells us that B^* is prime. B^* cannot be $\top_{n-1}$ (Theorem 87 and Theorem 80) and so, by Definition 147, B^* is an inverted bipartition.

There is another way to think of this. We can prove the theorem by induction. The definition of dual (in Theorem 72) tells us that it is true in C_0 as there is only one bipartition: $\top_0$ and one inverted bipartition: $\bot_0$. Let us assume that the theorem holds in C_k and B is bipartition in C_{k+1}. B is of the form $Dum_n(A)$ or $Vet_n(A)$ where A is a bipartition (Theorem 171). Theorem 101 and Theorem 97 tell us that the dual of this will be $Dum_n(A^*)$ or $Pas_n(A^*)$. The induction hypothesis tells us that A^* is an inverted bipartition. Theorem 173 tells us that $Dum_n(A^*)$ and $Pas_n(A^*)$ are inverted bipartitions.

□

Definition 176. Every bipartition B of C_{n+1} is either of the form $Dum_n(D)$ or $Vet_n(D)$ where, D is a bipartition (Theorem 171) of C_n.

I define *the complement of B* by recursion.

If $B = Dum_n(D)$ then $B^c = Vet_n(D^c)$

If $B = Vet_n(D)$ then $B^c = Dum_n(D^c)$

In C_0, there is only one bipartition:$\top_0$. I define $\top_0^c$ to be $\top_0$.

Comments 177. We can see from Comment 172 that this definition makes sense. $\top_0$ is the bipartition where nobody votes 'yes' in a world without any voters and so it is its own complement.

Theorem 178. *If B is a bipartition then* $(B^c)^c = B$

Proof. In C_0, we have, $(\top_0^c)^c = \top_0^c = \top_0$ By Definition 176.

Let us assume that the result is true in C_k. Let B be a bipartition of C_{k+1}. B must be of the form $Vet_k(D)$ or $Dum_k(D)$ with D a bipartition of C_k (Theorem 171)

First, let us assume that $B = Vet_k(D)$

$(B^c)^c = (Vet_k(D)^c)^c = Dum_k(D^c)^c = Vet_k((D^c)^c)$

$Vet_k((D^c)^c) = Vet_k(D)$ by the induction hypothesis.

Next, let us assume that $B = Dum_k(D)$

$(B^c)^c = (Dum_k(D)^c)^c = Vet_k(D^c)^c = Dum_k((D^c)^c)$

$Dum_k((D^c)^c) = Dum_k(D)$ by the induction hypothesis.

□

Definition 179. Given bipartitions: A and B, in C_n, the *bipartition conjunction: $A \wedge^b B$* is defined by recursion on n as follows:

$Vet_{n-1}(C) \wedge^b Vet_{n-1}(D) = Vet_{n-1}(C \wedge^b D)$

$Vet_{n-1}(C) \wedge^b Dum_{n-1}(D) = Dum_{n-1}(C \wedge^b D)$

$Dum_{n-1}(C) \wedge^b Vet_{n-1}(D) = Dum_{n-1}(C \wedge^b D)$

$Dum_{n-1}(C) \wedge^b Dum_{n-1}(D) = Dum_{n-1}(C \wedge^b D)$

In C_0 there is one bipartition: $\top_0$. $\top_0 \wedge^b \top_0 = \top_0$.

Comments 180. The n^{th} voter votes 'yes' in $A \wedge^b B$ if and only if it votes 'yes' in A and B

Definition 181. Given bipartitions: A and B, in C_n, the *bipartition disjunction: $A \vee^b B$* is defined by recursion on n as follows:

$Vet_{n-1}(C) \vee^b Vet_{n-1}(D) = Vet_{n-1}(C \vee^b D)$

$Vet_{n-1}(C) \vee^b Dum_{n-1}(D) = Vet_{n-1}(C \vee^b D)$

$Dum_{n-1}(C) \vee^b Vet_{n-1}(D) = Vet_{n-1}(C \vee^b D)$

$Dum_{n-1}(C) \vee^b Dum_{n-1}(D) = Dum_{n-1}(C \vee^b D)$

In C_0 there is one bipartition: $\top_0$. $\top_0 \vee^b \top_0 = \top_0$.

Comments 182. The n^{th} voter votes 'yes' in $A \vee^b B$ if and only if it votes 'yes' in A or B

Theorem 183. *For all bipartitions A in C_n, $A \wedge^b \top_n = \top_n$*

Proof. The proof is by induction. In C_0 it is true as there is only one bipartition: $\top_0$.

Let us assume that the theorem is true in C_k.

Consider A in C_{k+1}. A could be equal to $Dum_k(B)$ or $Vet_k(B)$ (Theorem 171).

If $A = Dum_k(B)$ then $A \wedge^b \top_{k+1} = Dum_k(B) \wedge^b Dum_k(\top_k)$ (Definition 85).

By Definition 179 this is equal to $Dum_k(B \wedge^b \top_k)$. The induction hypothesis tells us that this is $Dum(\top_k)$ which is $\top_{k+1}$ by Definition 85. This completes the induction

If $A = Vet_k(B)$ then $A \wedge^b \top_{k+1} = Vet_k(B) \wedge^b Dum_k(\top_k)$ (Definition 85).

By Definition 179 this is equal to $Dum_k(B \wedge^b \top_k)$.

The induction hypothesis tells us that this is $Dum_k(\top_k)$ which is $\top_{k+1}$ by Definition 85. This completes the induction

□

Comments 184. The conjunction of a bipartition with the bipartition in which everybody voters 'no' is equal to the bipartition in which everybody voters 'no'.

Theorem 185. *For all bipartitions A in C_n, $A \vee^b \top_n = A$.*

Proof. The proof is by induction. In C_0 it is true as there is only one bipartition: $\top_0$.

Let us assume that the theorem is true in C_k.

Consider A in C_{k+1}. A could be equal to $Dum_k(B)$ or $Vet_k(B)$ (Theorem 171).

If $A = Dum_k(B)$ then $A \vee^b \top_{k+1} = Dum_k(B) \vee^b Dum_k(\top_k)$ (Definition 85).

By Definition 181, this is equal to $Dum_k(B \vee^b \top_k)$.

The induction hypothesis tells us that this is $Dum_k(B)$, which is equal to A.

If $A = Vet_k(B)$ then $A \vee^b \top_{k+1} = Vet_k(B) \vee^b Dum_k(\top_k)$ (Definition 85).

By Definition 181, this is equal to $Vet_k(B \vee^b \top_k)$.

The induction hypothesis tells us that this is $Vet_k(B)$, which is equal to A.

□

Comments 186. The disjunction of a bipartition with the bipartition in which everybody voters 'no' is equal to the original bipartition.

Theorem 187. *If B and D are bipartitions then $(B \wedge^b D)^c = B^c \vee^b D^c$ and $(B \vee^b D)^c = B^c \wedge^b D^c$*

Proof. We need $(\top_0 \wedge^b \top_0)^c = \top_0^c \vee^b \top_0^c$.

This must be true because there is only one bipartition in C_0:$\top_0$.

Now let us assume that the result holds in C_k. Let B and D be objects of C_{k+1}. B can be of the form $Vet_k(E)$ and $Dum_k(E)$ with E a bipartition (Theorem 171). For similar reasons. I will write D as $Vet_k(F)$ or $Dum_k(F)$.

If $B = Vet_k(E)$ and $D = Vet_k(F)$ then:

$(B \wedge^b D)^c$

$= (Vet_k(E) \wedge^b Vet_k(F))^c$

$= (Vet_k(E \wedge^b F))^c$ (Definition 179)

$= Dum_k((E \wedge^b F)^c))$ (Definition 176)

$= Dum_k(E^c \vee^b F^c)$ by the induction hypothesis.

$= Dum_k(E^c) \vee^b Dum_k(F^c)$ (Definition 181)

$= Vet_k(E)^c \vee^b Vet_k(F)^c$ (Definition 176)

$= B^c \vee^b D^c$.

If $B = Vet_k(E)$ and $D = Dum_k(F)$ then:

$(B \wedge^b D)^c$

$= (Vet_k(E) \wedge^b Dum_k(F))^c$

$= (Dum_k(E \wedge^b F))^c$ (Definition 179)

$= Vet_k((E \wedge^b F)^c))$ (Definition 176)

$= Vet_k(E^c \vee^b F^c)$ by the induction hypothesis.

$= Dum_k(E^c) \vee^b Vet_k(F^c)$ (Definition 181)

$= Vet_k(E)^c \vee^b Dum_k(F)^c$ (Definition 176)

$= B^c \vee^b D^c$.

If $B = Dum_k(E)$ and $D = Vet_k(F)$ then:

$(B \wedge^b D)^c$

$= (Dum_k(E) \wedge^b Vet_k(F))^c$

$= (Dum_k(E \wedge^b F))^c$ (Definition 179)

$= Vet_k((E \wedge^b F)^c))$ (Definition 176)

$= Vet_k(E^c \vee^b F^c)$ by the induction hypothesis.

$= Vet_k(E^c) \vee^b Dum_k(F^c)$ (Definition 181)

$= Dum_k(E)^c \vee^b Vet_k(F)^c$ (Definition 176)

$= B^c \vee^b D^c$.

If $B = Dum_k(E)$ and $D = Dum_k(F)$ then:

$(B \wedge^b D)^c$

$= (Dum_k(E) \wedge^b Dum_k(F))^c$

$= (Dum_k(E \wedge^b F))^c$ (Definition 179)

$= Vet_k((E \wedge^b F)^c))$ (Definition 176)

$= Vet_k(E^c \vee^b F^c)$ by the induction hypothesis.

$= Vet_k(E^c) \vee^b Vet_k(F^c)$ (Definition 181)

$= Dum_k(E)^c \vee^b Dum_k(F)^c$ (Definition 176)

$= B^c \vee^b D^c$.

$\square$

Theorem 188. *$A \wedge^b B = A$ iff $A \vee^b B = B$*

Proof. In C_0 there is one bipartition:$\top_0$. $\top_0 \wedge^b \top_0 = \top_0$ and $\top_0 \vee^b \top_0 = \top_0$ (Definitions 179 and 181)

Let us assume that the theorem holds in C_k.

Theorem 171 tells us that A can be of the form $Vet_k(C)$ or $Dum_k(C)$ with C a bipartition and B can be of the form $Vet_k(E)$ or $Dum_k(E)$ with E a bipartition.

For the induction step, first let us say that $A = Dum_k(C)$ and $B = Dum_k(E)$.

$A \wedge_b B = A$

$\iff Dum_k(C) \wedge^b Dum_k(E) = Dum_k(C)$

$\iff Dum_k(C \wedge^b E) = Dum_k(C)$ (Definition 179)

$\iff C \wedge^b E = C$ (Definition 83)

$\iff C \vee^b E = E$ (By the induction hypothesis)

$\iff Dum_k(C \vee^b E) = Dum_k(E)$ (By Definition 83)

$\iff Dum_k(C) \vee^b Dum_k(E) = Dum_k(E)$ (By Definition 181)

$A \vee^b B = B$

Next, let us say that $A = Vet_k(C)$ and $B = Dum_k(E)$

$A \wedge_b B = A$

$\iff Vet_k(C) \wedge^b Dum_k(E) = Vet_k(C)$

$\iff Dum_k(C \wedge^b E) = Vet_k(C)$ (Definition 179)

This is impossible by Definition 83 and Definition 88 and the fact that $\perp_0$ is not a bipartition while $C \wedge^b E$ is.

$A \vee^b B = B$

$\iff Vet_k(C) \vee^b Dum_k(E) = Dum_k(E)$

$\iff Vet_k(C \vee^b E) = Dum_k(E)$ (Definition 181

This is impossible by Definition 83 and Definition 88 and the fact that $\perp_0$ is not a bipartition while $C \vee^b E$ is.

So neither of these are possible.

Next let us say that $A = Dum_k(C)$ and $B = Vet_k(E)$ with A and B objects of C_{k+1}.

$A \wedge_b B = A$

$\iff Dum_k(C) \wedge^b Vet_k(E) = Dum_k(C)$

$\iff Dum_k(C \wedge^b E) = Dum_k(C)$ (Definition 179)

$\iff C \wedge^b E = C$ (Definition 83)

$\iff C \vee^b E = E$ (By the induction hypothesis)

$\iff Vet_k(C \vee^b E) = Vet_k(E)$ (By Definition 88)

$\iff Dum_k(C) \vee^b Vet_k(E) = Vet_k(E)$ (By Definition 181)

$A \vee^b B = B$

Finally, let us say that $A = Vet_k(C)$ and $B = Vet_k(E)$.

$A \wedge_b B = A$

$\iff Vet_k(C) \wedge^b Vet_k(E) = Vet_k(C)$

$\iff Vet_k(C \wedge^b E) = Vet_k(C)$ (Definition 179)

$\iff C \wedge^b E = C$ (Definition 88)

$\iff C \vee^b E = E$ (By the induction hypothesis)

$\iff Vet_k(C \vee^b E) = Vet_k(E)$ (By Definition 88)

$\iff Vet_k(C) \vee^b Vet_k(E) = Vet_k(E)$ (By Definition 181)

$A \vee^b B = B$

This covers all the cases and so completes the induction.

□

Definition 189. By Theorem 188 $A \wedge^b B = A$ iff $A \vee^b B = B$. In both cases, we say that $A \leq^b B$.

Theorem 190. *Every bipartition of C_n is of the form $Dum_n(C)$ or $Vet_n(C)$ with C a bipartition of C_{n-1} (Theorem 171). If $A \leq^b B$ and A is of the form $Vet_n(C)$ then B is of the form $Vet_n(D)$ where D is also a bipartition of C_{n-1}.*

Proof. Let us say that $Vet_n(C) \leq^b Dum_n(D)$.

Definition 189 tells us that $Vet_n(C) \vee^b Dum_n(D) = Dum_n(D)$.

Definition 181 then gives us $Vet_n(C \vee^b D) = Dum_n(D)$.

Definition 88 and Definition 83 now tell us that $Dom_{n+1}(Vet_n(C \vee^b D)) = \perp_k$ and $Dom_{n+1}(Dum_n(D)) = D$. But $\perp_k$ is not a bipartition (Definition 145). This contradiction shows us that $Vet_n(C \vee^b D) \leq^b Dum_n(D)$ was not possible.

□

Comments 191. This says that if v_n votes 'yes' in A and B is greater than A then v_n also votes 'yes' in B.

Theorem 192. *Let A and B be bipartitions of C_n.*

$Dum_n(A) \leq^b Dum_n(B) \iff A \leq^b B$

$Dum_n(A) \leq^b Vet_n(B) \iff A \leq^b B$

$Vet_n(A) \leq^b Vet_n(B) \iff A \leq^b B$

Of course these are the only possibilities (Theorem 171 and Theorem 190)

Proof. $Dum(A) \leq^b Dum(B)$

$\iff Dum(A) = Dum(A) \wedge^b Dum(B) \iff$ (Definition 189)

$\iff Dum(A) = Dum(A \wedge^b B)$ (Definition 179)

$\iff A = A \wedge^b B \iff$ (Definition 83)

$\iff A \leq^b B$ (Definition 189)

$Dum(A) \leq^b Vet(B)$

$\iff Dum(A) = Dum(A) \wedge^b Vet(B)$ (Definition 189)

$\iff Dum(A) = Dum(A \wedge^b B)$ (Definition 179)

$\iff A = A \wedge^b B$ (Definition 83)

$\iff A \leq^b B$ (Definition 189)

$Vet(A) \leq^b Vet(B)$

$\iff Vet(A) = Vet(A) \wedge^b Vet(B)$ (Definition 189)

$\iff Vet(A) = Vet(A \wedge^b B)$ (Definition 179)

$\iff A = A \wedge^b B \iff$ (Definition 88)

$\iff A \leq^b B$ (Definition 189).

□

Theorem 193. $\leq^b$ *is reflexive on all bipartitions in C_n*

Proof. The proof is by induction on n. In C_0 there is one bipartition: $\top_0$. $\top_0 \wedge^b \top_0 = \top_0$ and so $\top_0 \leq^b \top_0$ (Definition 189).

Let us assume that the theorem is true in C_k.

Let B be a bipartition in C_k. It is either of the form $Dum_k(A)$ or $Vet_k(A)$ with A a bipartition (Theorem 171).

If $B = Dum_k(A)$ then

$B \leq^b B \iff$

$Dum_k(A) \leq^b Dum_k(A) \iff$ Using Theorem 192

$A \leq^b A$ Which is true by the induction hypothesis

If $B = Vet_k(A)$ then

$B \leq^b B \iff$

$Vet_k(A) \leq^b Vet_k(A) \iff$ Using Theorem 192

$A \leq^b A$ Which is true by the induction hypothesis

Having covered both cases, we have completed the induction.

□

Theorem 194. *Let A and B be bipartitions of C_n. $A \leq^b B$ and $B \leq^b A$ imply $A = B$.*

Proof. The proof is by induction. In C_0 it is true because there is only one bipartition: $\top_0$.

Let us assume that it is true in C_k, let A and B be bipartitions in C_{k+1}.

Theorem 171 tells us that A and B are of the form $Vet_k(C)$ or $Dum_k(C)$ with C a bipartition.

Theorem 190 tells us that A and B are both of the same form.

Let us assume that $A = Dum_k(C)$ and $B = Dum_k(D)$

$A \leq^b B$ and $B \leq^b A \iff$

$Dum_k(C) \leq^b Dum_k(D)$ and $Dum_k(D) \leq^b Dum_k(C)$

$\iff C \leq^b D$ and $D \leq^b C \iff$ (Theorem 192)

$\iff C = D$ (Using the Induction Hypothesis)

$\iff Dum_k(C) = Dum_k(D)$ (Definition 83)

$A = B$ And we have completed the induction

Let us assume that $A = Vet_k(C)$ and $B = Vet_k(D)$

$A \leq^b B$ and $B \leq^b A \iff$

$Vet_k(C) \leq^b Vet_k(D)$ and $Vet_k(D) \leq^b Vet_k(C) \iff$ Using Theorem 192

$C \leq^b D$ and $D \leq^b C$

$\iff C = D$ (Using the Induction Hypothesis)

$\iff Vet_k(C) = Vet_k(D)$ (Definition 88)

$A = B$ And we have completed the induction

□

Theorem 195. *$\leq^b$ is transitive. That is, if A, B and C are bipartitions in C_n then $A \leq^b B$ and $B \leq^b C$ imply $A \leq^b C$*

Proof. The proof is by induction. It is true in C_0 because there is only one bipartition: $\top_0$.

Let us say that we have the result for C_k and A, B and C are bipartitions in C_{k+1}.

A is equal to $Dum_k(E)$ or $Vet_k(E)$ with E a bipartition (Theorem 171)

In the same way, B is equal to $Dum_k(F)$ or $Vet_k(F)$ with F a bipartition and C is equal to $Dum_k(G)$ or $Vet_k(G)$ with G a bipartition.

First, let us assume that A is of the form $Vet_k(E)$ (with E a bipartition). Theorem 190 tells us that B and C are of the form $Vet_k(F)$ and $Vet_k(G)$.

$A \leq^b B$ and $B \leq^b C$

$Vet_k(D) \leq^b Vet_k(E)$ and $Vet_k(E) \leq^b Vet_k(F)$

$\iff$ $Vet(D) = Vet(D) \wedge^b Vet(E)$ and $Vet(E) = Vet(E) \wedge^b Vet(F)$ (Definition 189)

$\iff$ $Vet(D) = Vet(D \wedge^b E)$ and $Vet(E) = Vet(E \wedge^b F)$ (Definition 179)

$\iff$ $D = D \wedge^b E$ and $E = E \wedge^b F$ (Definition 88)

$\iff$ $D \leq^b E$ and $E \leq^b F$ (Definition 189)

$\implies$ $D \leq^b F$ (Induction Hypothesis)

$\iff$ $D = D \wedge^b F$ (Definition 189)

$\iff$ $Vet(D) = Vet(D \wedge^b F)$ (Definition 88)

$\iff$ $Vet(D) = Vet(D) \wedge^b Vet(F)$ (Definition 179)

$\iff$ $Vet(D) \leq^b Vet(F)$ (Definition 189)

$\iff$ $A \leq^b C$.

Next, let us assume that A is of the form $Dum(D)$ (with D a bipartition). B can be of the form $Dum(E)$ or $Vet(E)$. Let us assume that it is $Vet(E)$. Theorem 190 tell us that C is of the form $Vet(F)$.

$A \leq^b B$ and $B \leq^b C$

$Dum_k(D) \leq^b Vet_k(E)$ and $Vet_k(E) \leq^b Vet_k(F)$

$\iff$ $Dum(D) = Dum(D) \wedge^b Vet(E)$ and $Vet(E) = Vet(E) \wedge^b Vet(F)$ (Definition 189)

$\iff$ $Dum(D) = Dum(D \wedge^b E)$ and $Vet(E) = Vet(E \wedge^b F)$ (Definition 179)

$\iff$ $D = D \wedge^b E$ and $E = E \wedge^b F$ (Definition 88 and Definition 83)

$\iff$ $D \leq^b E$ and $E \leq^b F$ (Definition 189)

$\implies$ $D \leq^b F$ (Induction Hypothesis)

$\iff$ $D = D \wedge^b F$ (Definition 189)

$\iff$ *Dum*$(D) =$ *Dum*$(D \wedge^b F)$ (Definition 83)

$\iff$ *Dum*$(D) =$ *Dum*$(D) \wedge^b$ *Vet*(F) (Definition 179)

$\iff$ *Dum*$(D) \leq^b$ *Vet*(F) (Definition 189)

$\iff A \leq^b C$.

Next, let us assume that A is of the form *Dum*(D) (with D a bipartition) and B is of the form *Dum*(E) (with E a bipartition). C can be of the form *Dum*(F) or *Vet*(F). In this case, let us assume that it is *Vet*(F).

$A \leq^b B$ and $B \leq^b C$

$Dum_k(D) \leq^b Dum_k(E)$ and $Dum_k(E) \leq^b Vet_k(F)$

$\iff$ *Dum*$(D) =$ *Dum*$(D) \wedge^b$ *Dum*(E) and *Dum*$(E) =$ *Dum*$(E) \wedge^b$ *Vet*(F) (Definition 189)

$\iff$ *Dum*$(D) =$ *Dum*$(D \wedge^b E)$ and *Dum*$(E) =$ *Dum*$(E \wedge^b F)$ (Definition 179)

$\iff D = D \wedge^b E$ and $E = E \wedge^b F$ (Definition 83)

$\iff D \leq^b E$ and $E \leq^b F$ (Definition 189)

$\implies D \leq^b F$ (Induction Hypothesis)

$\iff D = D \wedge^b F$ (Definition 189)

$\iff$ *Dum*$(D) =$ *Dum*$(D \wedge^b F)$ (Definition 83)

$\iff$ *Dum*$(D) =$ *Dum*$(D) \wedge^b$ *Vet*(F) (Definition 179)

$\iff$ *Dum*$(D) \leq^b$ *Vet*(F) (Definition 189)

$\iff A \leq^b C$.

The last case has A of the form *Dum*(D), B of the form *Dum*(E) and C of the form *Dum*(F).

$A \leq^b B$ and $B \leq^b C$

$Dum_k(D) \leq^b Dum_k(E)$ and $Dum_k(E) \leq^b Dum_k(F)$

$\iff$ $Dum(D) = Dum(D) \wedge^b Dum(E)$ and $Dum(E) = Dum(E) \wedge^b Dum(F)$ (Definition 189)

$\iff$ $Dum(D) = Dum(D \wedge^b E)$ and $Dum(E) = Dum(E \wedge^b F)$ (Definition 179)

$\iff$ $D = D \wedge^b E$ and $E = E \wedge^b F$ (Definition 83)

$\iff$ $D \leq^b E$ and $E \leq^b F$ (Definition 189)

$\implies$ $D \leq^b F$ (Induction Hypothesis)

$\iff$ $D = D \wedge^b F$ (Definition 189)

$\iff$ $Dum(D) = Dum(D \wedge^b F)$ (Definition 83)

$\iff$ $Dum(D) = Dum(D) \wedge^b Dum(F)$ (Definition 179)

$\iff$ $Dum(D) \leq^b Dum(F)$ (Definition 189)

$\iff$ $A \leq^b C$.

These are all the possibilities and so this completes the proof.

$\square$

Theorem 196. *$\leq^b$ is a partial order on the set of all bipartitions in C_n.*

Proof. $\leq^b$ is reflexive (Theorem 193), antisymmetric (Theorem 194) and transitive (Theorem 195).

$\square$

Theorem 197. *If A is a bipartition of C_n then $Dum_n(A) \leq^b Vet_n(A)$*

Proof. $Dum_n(A) \leq^b Vet_n(A)$

$\iff$ $Dum_n(A) = Dum_n(A) \wedge^b Vet_n(A)$ (Definition 189)

$\iff$ $Dum_n(A) = Dum_n(A \wedge^b A)$ (Definition 179)

$\iff$ $A = A \wedge^b A$ (Definition 83)

Which is true by Theorem 136

$\square$

Theorem 198. *If $E \leq^b F$ then $F^c \leq^b E^c$*

Proof. $E \leq^b F$

$\iff E = E \wedge^b F \iff$ (Definition 189)

$\iff E^c = E^c \vee^b F^c$ (Taking complements and using Theorem 187)

$\iff F^c \leq^b E^c$ (Definition 189)

□

Theorem 199. *If A and B are bipartitions in C_n, $A \vee^b B$ is the least upper bound of A and B with respect to $\leq^b$*

Proof. The proof is by induction on n. It is true in C_0. There is only one object in C_0.

Let us assume that the result holds in C_k. Let A and B be objects of C_{k+1}.

A must be of the form $Dum_k(C)$ or $Vet_k(C)$ and B must be of the form $Dum_k(D)$ or $Vet_k(D)$ (Theorem 171).

This gives us four cases:

Case one: $A = Dum_k(C)$ and $B = Dum_k(D)$.

First, I need $A \leq^b A \vee^b B$

$\iff Dum_k(C) \leq^b Dum_k(C) \vee^b Dum_k(D)$

$\iff Dum_k(C) = Dum_k(C) \wedge^b (Dum_k(C) \vee^b Dum_k(D))$ (Definition 189)

$\iff Dum_k(C) = Dum_k(C) \wedge^b Dum_k(C \vee^b D)$ (Definition 181)

$\iff Dum_k(C) = Dum_k(C \wedge^b (C \vee^b D))$ (Definition 179)

$\iff C = C \wedge^b (C \vee^b D)$ (Definition 83)

$\iff C \leq^b C \vee^b D$ (Definition 189)

Which is true by the induction hypothesis.

Obviously, the proof that $B \leq^b A \vee^b B$ is similar.

Next, let us assume that $A \leq^b E$ and $B \leq^b E$. I need to show that $A \vee^b B \leq^b E$. E can be $Dum_k(F)$ or $Vet_k(F)$.

First, say $E = Dum_k(F)$.

$A \leq^b Dum_k(F)$ and $B \leq^b Dum_k(F)$

$\iff$ $Dum_k(C) \leq^b Dum_k(F)$ and $Dum_k(D) \leq^b Dum_k(F)$

$\iff$ $Dum_k(F) = Dum_k(C) \vee^b Dum_k(F)$ and $Dum_k(F) = Dum_k(D) \vee^b Dum_k(F)$ (Definition 189)

$\iff$ $Dum_k(F) = Dum_k(C \vee^b F)$ and $Dum_k(F) = Dum_k(D \vee^b F)$ (Definition 181)

$\iff$ $F = C \vee^b F$ and $F = D \vee^b F$ (Definition 83)

$\iff$ $C \leq^b F$ and $D \leq^b F$ (Definition 189)

$\iff$ $C \vee^b D \leq^b F$ (By the induction hypothesis)

$\iff$ $Dum_k(C \vee^b D) \leq^b Dum_k(F)$ (Definition 83)

$\iff$ $Dum_k(C) \vee^b Dum_k(D) \leq^b Dum_k(F)$ (By Definition 181)

$\iff$ $A \vee^b B \leq^b Dum_k(F)$

Next, say $E = Vet_k(F)$.

$A \leq^b Vet_k(F)$ and $B \leq^b Vet_k(F)$

$\iff$ $Dum_k(C) \leq^b Vet_k(F)$ and $Dum_k(D) \leq^b Vet_k(F)$

$\iff$ $Vet_k(F) = Dum_k(C) \vee^b Vet_k(F)$ and $Vet_k(F) = Dum_k(D) \vee^b Vet_k(F)$ (Definition 189)

$\iff$ $Vet_k(F) = Vet_k(C \vee^b F)$ and $Vet_k(F) = Vet_k(D \vee^b F)$ (Definition 181)

$\iff$ $F = C \vee^b F$ and $F = D \vee^b F$ (Definition 88)

$\iff$ $C \leq^b F$ and $D \leq^b F$ (Definition 189)

$\iff$ $C \vee^b D \leq^b F$ (By the induction hypothesis)

$\iff$ $Vet_k(C \vee^b D) \leq^b Vet_k(F)$ (Definition 88)

$\implies$ $Dum_k(C \vee^b D) \leq^b Vet_k(F)$ (Theorem 197 and Theorem 195)

$\iff$ $Dum_k(C) \vee^b Dum_k(D) \leq^b Vet_k(F)$ (By Definition 181)

$\iff$ $A \vee^b B \leq^b Vet_k(F)$

$\iff$ $A \vee^b B \leq^b E$

Which completes the induction

Case two: $A = Dum_k(C)$ and $B = Vet_k(D)$.

First, I need $A \leq^b A \vee^b B$

$\iff$ $Dum_k(C) \leq^b Dum_k(C) \vee^b Vet_k(D)$

$\iff$ $Dum_k(C) = Dum_k(C) \wedge^b (Dum_k(C) \vee^b Vet_k(D))$ (Definition 189)

$\iff$ $Dum_k(C) = Dum_k(C) \wedge^b Vet_k(C \vee^b D)$ (Definition 181)

$\iff$ $Dum_k(C) = Dum_k(C \wedge^b (C \vee^b D))$ (Definition 179)

$\iff$ $C = C \wedge^b (C \vee^b D)$ (Definition 83)

$\iff$ $C \leq^b C \vee^b D$ (Definition 189)

Which is true by the induction hypothesis.

Obviously, the proof that $B \leq^b A \vee^b B$ is similar.

Next, let us assume that $A \leq^b E$ and $B \leq^b E$. I need to show that $A \vee^b B \leq^b E$. E can be $Dum_k(F)$ or $Vet_k(F)$.

First, say $E = Dum_k(F)$.

$A \leq^b Dum_k(F)$ and $B \leq^b Dum_k(F)$ cannot be true by Theorem 190

Next, say $E = Vet_k(F)$.

$A \leq^b Vet_k(F)$ and $B \leq^b Vet_k(F)$

$\iff$ $Dum_k(C) \leq^b Vet_k(F)$ and $Vet_k(D) \leq^b Vet_k(F)$

$\iff$ $Vet_k(F) = Dum_k(C) \vee^b Vet_k(F)$ and $Vet_k(F) = Vet_k(D) \vee^b Vet_k(F)$ (Definition 189)

$\iff$ $Vet_k(F) = Vet_k(C \vee^b F)$ and $Vet_k(F) = Vet_k(D \vee^b F)$ (Definition 181)

$\iff$ $F = C \vee^b F$ and $F = D \vee^b F$ (Definition 88)

$\iff$ $C \leq^b F$ and $D \leq^b F$ (Definition 189)

$\iff$ $C \vee^b D \leq^b F$ (By the induction hypothesis)

$\iff$ $Vet_k(C \vee^b D) \leq^b Vet_k(F)$ (Definition 88)

$\iff$ $Dum_k(C) \vee^b Vet_k(D) \leq^b Vet_k(F)$ (By Definition 181)

$\iff$ $A \vee^b B \leq^b Vet_k(F)$

$\iff$ $A \vee^b B \leq^b E$

Case three: $A = Vet_k(C)$ and $B = Dum_k(D)$.

First, I need $A \leq^b A \vee^b B$

$\iff$ $Vet_k(C) \leq^b Vet_k(C) \vee^b Dum_k(D)$

$\iff$ $Vet_k(C) = Vet_k(C) \wedge^b (Vet_k(C) \vee^b Dum_k(D))$ (Definition 189)

$\iff$ $Vet_k(C) = Vet_k(C) \wedge^b Vet_k(C \vee^b D)$ (Definition 181)

$\iff$ $Vet_k(C) = Vet_k(C \wedge^b (C \vee^b D))$ (Definition 179)

$\iff$ $C = C \wedge^b (C \vee^b D)$ (Definition 83)

$\iff$ $C \leq^b C \vee^b D$ (Definition 189)

Which is true by the induction hypothesis.

Obviously, the proof that $B \leq^b A \vee^b B$ is similar.

Next, let us assume that $A \leq^b E$ and $B \leq^b E$. I need to show that $A \vee^b B \leq^b E$. E can be $Dum_k(F)$ or $Vet_k(F)$.

First, say $E = Dum_k(F)$.

$A \leq^b Dum_k(F)$ and $B \leq^b Dum_k(F)$

Implies that $Vet_k(C) \leq^b Dum_k(F)$ This is not possible by Theorem 190.

Next, say $E = Vet_k(F)$.

$A \leq^b Vet_k(F)$ and $B \leq^b Vet_k(F)$

$\iff Vet_k(C) \leq^b Vet_k(F)$ and $Dum_k(D) \leq^b Vet_k(F)$

$\iff Vet_k(F) = Vet_k(C) \vee^b Vet_k(F)$ and $Vet_k(F) = Dum_k(D) \vee^b Vet_k(F)$ (Definition 189)

$\iff Vet_k(F) = Vet_k(C \vee^b F)$ and $Vet_k(F) = Vet_k(D \vee^b F)$ (Definition 181)

$\iff F = C \vee^b F$ and $F = D \vee^b F$ (Definition 88)

$\iff C \leq^b F$ and $D \leq^b F$ (Definition 189)

$\iff C \vee^b D \leq^b F$ (By the induction hypothesis)

$\iff Vet_k(C \vee^b D) \leq^b Vet_k(F)$ (Definition 88)

$\iff Vet_k(C) \vee^b Dum_k(D) \leq^b Vet_k(F)$ (By Definition 181)

$\iff A \vee^b B \leq^b Dum_k(F)$

$\iff A \vee^b B \leq^b E$.

Case four: $A = Vet_k(C)$ and $B = Vet_k(D)$.

First, I need $A \leq^b A \vee^b B$

$\iff Vet_k(C) \leq^b Vet_k(C) \vee^b Vet_k(D)$

$\iff Vet_k(C) = Vet_k(C) \wedge^b (Vet_k(C) \vee^b Vet_k(D))$ (Definition 189)

$\iff Vet_k(C) = Vet_k(C) \wedge^b Vet_k(C \vee^b D)$ (Definition 181)

$\iff Vet_k(C) = Vet_k(C \wedge^b (C \vee^b D))$ (Definition 179)

$\iff C = C \wedge^b (C \vee^b D)$ (Definition 83)

$\iff C \leq^b C \vee^b D$ (Definition 189)

Which is true by the induction hypothesis.

Obviously, the proof that $B \leq^b A \vee^b B$ is similar.

Next, let us assume that $A \leq^b E$ and $B \leq^b E$. I need to show that $A \vee^b B \leq^b E$. E can be $Dum_k(F)$ or $Vet_k(F)$.

First, say $E = Dum_k(F)$.

$A \leq^b Dum_k(F)$ and $B \leq^b Dum_k(F)$

Would suggest that $Vet_k(C) \leq^b Dum_k(F)$ This is not possible by Theorem 190

Next, say $E = Vet_k(F)$.

$A \leq^b Vet_k(F)$ and $B \leq^b Vet_k(F)$

$\iff$ $Vet_k(C) \leq^b Vet_k(F)$ and $Vet_k(D) \leq^b Vet_k(F)$

$\iff$ $Vet_k(F) = Vet_k(C) \vee^b Vet_k(F)$ and $Vet_k(F) = Vet_k(D) \vee^b Vet_k(F)$ (Definition 189)

$\iff$ $Vet_k(F) = Vet_k(C \vee^b F)$ and $Vet_k(F) = Vet_k(D \vee^b F)$ (Definition 181)

$\iff$ $F = C \vee^b F$ and $F = D \vee^b F$ (Definition 88)

$\iff$ $C \leq^b F$ and $D \leq^b F$ (Definition 189)

$\iff$ $C \vee^b D \leq^b F$ (By the induction hypothesis)

$\iff$ $Vet_k(C \vee^b D) \leq^b Vet_k(F)$ (Definition 88)

$\iff$ $Vet_k(C) \vee^b Vet_k(D) \leq^b Vet_k(F)$ (By Definition 181)

$\iff$ $A \vee^b B \leq^b Vet_k(F)$

$\iff$ $A \vee^b B \leq^b E$

$\square$

Theorem 200. *$A \wedge^b B$ is the greatest lower bound of A and B with respect to $\leq^b$*

Proof. This is the dual of Theorem 199.

$\square$

Theorem 201. *$\top_n$ is the minimal bipartition in C_n relative to $\leq^b$*

Proof. For all bipartitions A, $A \vee^b \top_n = A$ by Theorem 185 and so $\top_n \leq^b A$ (By Definition 189)

□

Theorem 202. *The set of bipartitions in C_n with two binary operations: bipartition conjunction ($\wedge^b$) and bipartition disjunction ($\vee^b$) and two defined objects $\top_n$ (as the 0 in the BA) and its complement: $\top_n^c$ (as the 1 in the BA) constitute a Boolean Algebra*

Proof. I will tackle the axioms one by one.

First, I need to show that both operations are commutative.

$E \vee^b F = F \vee^b E$

The easiest way to see this is to recognise that $E \vee^b F$ is the least upper bound of E and F. This operation is independent of order. It is also clear that $\vee^b$ is commutative from Definition 181 and a simple induction as follows.

It is clearly true in C_0; there is only one bipartition in C_0.

E, in C_n can be $Dum_n(C)$ or $Vet_n(C)$ with C a bipartition.

F can be $Dum_n(D)$ or $Vet_n(D)$ with D a bipartition.

$Dum_n(C) \vee^b Dum_n(D)$

$= Dum_n(C \vee^b D)$ Using Definition 181

$= Dum_n(D \vee^b C)$ By the induction hypothesis

$= Dum_n(D) \vee^b Dum_n(C)$ Using Definition 181

$Dum_n(C) \vee^b Vet_n(D)$

$= Vet_n(C \vee^b D)$ Using Definition 181

$= Vet_n(D \vee^b C)$ By the induction hypothesis

$= Vet_n(D) \vee^b Dum_n(C)$ Using Definition 181

$Vet_n(C) \vee^b Dum_n(D)$

$= \mathit{Vet}_n(C \vee^b D)$ Using Definition 181

$= \mathit{Vet}_n(D \vee^b C)$ By the induction hypothesis

$= \mathit{Dum}_n(D) \vee^b \mathit{Vet}(C)$ Using Definition 181

$\mathit{Vet}_n(C) \vee^b \mathit{Vet}_n(D)$

$= \mathit{Vet}_n(C \vee^b D)$ Using Definition 181

$= \mathit{Vet}_n(D \vee^b C)$ By the induction hypothesis

$= \mathit{Vet}_n(D) \vee^b \mathit{Vet}_n(C)$ Using Definition 181

$E \wedge^b F = F \wedge^b E$

The easiest way to see this is to recognise that $E \wedge^b F$ is the greatest lower bound of E and F. This operation is independent of order. This is also just the dual of the previous result.

Next I need to show that the operations are associative

$(A \vee^b B) \vee^b C = A \vee^b (B \vee^b C)$

Again, I could just say that $\vee^b$ is the least upper bound and both sides are just the least upper bound of the three - it doesn't matter what order you take it in.

The rigorous proof is again by induction.

In C_0 it is obvious because there is only one bipartition: $\top_0$.

A is equal to $\mathit{Dum}_n(E)$ or $\mathit{Vet}_n(E)$ B is equal to $\mathit{Dum}_n(F)$ or $\mathit{Vet}_n(F)$ and C is equal to $\mathit{Dum}_n(G)$ or $\mathit{Vet}_n(G)$ (Theorem 171)

First, let us assume that $A = \mathit{Dum}_n(E)$, $B = \mathit{Dum}_n(F)$ and $C = \mathit{Dum}_n(G)$

$(\mathit{Dum}_n(E) \vee^b \mathit{Dum}_n(F)) \vee^b \mathit{Dum}_n(G)$

$= \mathit{Dum}_n(E \vee^b F) \vee^b \mathit{Dum}_n(G)$ Applying Definition 181

$= \mathit{Dum}_n((E \vee^b F) \vee^b G)$ Applying Definition 181

$= Dum(E \vee^b (F \vee^b G))$ By the induction hypothesis

$= Dum(E) \vee^b Dum(F \vee^b G)$ Applying Definition 181

$= Dum(E) \vee^b (Dum(F) \vee^b Dum(G))$ Applying Definition 181

This is the result we are seeking

Let us assume that $A = Dum_n(E)$, $B = Dum_n(F)$ and $C = Vet_n(G)$

$(Dum_n(E) \vee^b Dum_n(F)) \vee^b Vet_n(G)$

$= Dum_n(E \vee^b F) \vee^b Vet_n(G)$ Applying Definition 181

$= Vet_n((E \vee^b F) \vee^b G)$ Applying Definition 181

$= Vet_n(E \vee^b (F \vee^b G))$ By the induction hypothesis

$= Dum_n(E) \vee^b Vet_n(F \vee^b G)$ Applying Definition 181

$= Dum_n(E) \vee^b (Dum_n(F) \vee^b Vet_n(G))$ Applying Definition 181

This is the result we are seeking

Let us assume that $A = Dum_n(E)$, $B = Vet_n(F)$ and $C = Dum_n(G)$

$(Dum_n(E) \vee^b Vet_n(F)) \vee^b Dum_n(G)$

$= Vet_n(E \vee^b F) \vee^b Dum_n(G)$ Applying Definition 181

$= Vet_n((E \vee^b F) \vee^b G)$ Applying Definition 181

$= Vet_n(E \vee^b (F \vee^b G))$ By the induction hypothesis

$= Dum_n(E) \vee^b Vet(F \vee^b G)$ Applying Definition 181

$= Dum_n(E) \vee^b (Vet_n(F) \vee^b Dum_n(G))$ Applying Definition 181

This is the result we are seeking

Let us assume that $A = Dum_n(E)$, $B = Vet_n(F)$ and $C = Vet_n(G)$

$(Dum_n(E) \vee^b Vet_n(F)) \vee^b Vet_n(G)$

$= Vet_n(E \vee^b F) \vee^b Vet_n(G)$ Applying Definition 181

$= Vet_n((E \vee^b F) \vee^b G)$ Applying Definition 181

$= Vet_n(E \vee^b (F \vee^b G))$ By the induction hypothesis

$= Dum_n(E) \vee^b Vet(F \vee^b G)$ Applying Definition 181

$= Dum_n(E) \vee^b (Vet_n(F) \vee^b Vet_n(G))$ Applying Definition 181

This is the result we are seeking

Let us assume that $A = Vet_n(E)$, $B = Dum_n(F)$ and $C = Dum_n(G)$

$(Vet_n(E) \vee^b Dum_n(F)) \vee^b Dum_n(G)$

$= Vet_n(E \vee^b F) \vee^b Dum_n(G)$ Applying Definition 181

$= Vet_n((E \vee^b F) \vee^b G)$ Applying Definition 181

$= Vet_n(E \vee^b (F \vee^b G))$ By the induction hypothesis

$= Vet_n(E) \vee^b Dum(F \vee^b G)$ Applying Definition 181

$= Vet_n(E) \vee^b (Dum(F) \vee^b Dum(G))$ Applying Definition 181

This is the result we are seeking

Let us assume that $A = Vet_n(E)$, $B = Dum_n(F)$ and $C = Vet_n(G)$

$(Vet_n(E) \vee^b Dum_n(F)) \vee^b Vet_n(G)$

$= Vet_n(E \vee^b F) \vee^b Vet_n(G)$ Applying Definition 181

$= Vet_n((E \vee^b F) \vee^b G)$ Applying Definition 181

$= Vet(E \vee^b (F \vee^b G))$ By the induction hypothesis

$= Vet(E) \vee^b Vet(F \vee^b G)$ Applying Definition 181

$= Vet(E) \vee^b (Dum(F) \vee^b Vet(G))$ Applying Definition 181

This is the result we are seeking

Let us assume that $A = Vet_n(E)$, $B = Vet_n(F)$ and $C = Dum_n(G)$

$(Vet_n(E) \vee^b Vet_n(F)) \vee^b Dum_n(G)$

$= Vet_n(E \vee^b F) \vee^b Dum_n(G)$ Applying Definition 181

$= Vet_n((E \vee^b F) \vee^b G)$ Applying Definition 181

$= Vet_n(E \vee^b (F \vee^b G))$ By the induction hypothesis

$= Vet_n(E) \vee^b Vet(F \vee^b G)$ Applying Definition 181

$= Vet_n(E) \vee^b (Vet_n(F) \vee^b Dum_n(G))$ Applying Definition 181

This is the result we are seeking

Let us assume that $A = Vet_n(E)$, $B = Vet_n(F)$ and $C = Vet_n(G)$

$(Vet_n(E) \vee^b Vet_n(F)) \vee^b Vet_n(G)$

$= Vet_n(E \vee^b F) \vee^b Vet_n(G)$ Applying Definition 181

$= Vet_n((E \vee^b F) \vee^b G)$ Applying Definition 181

$= Vet_n(E \vee^b (F \vee^b G))$ By the induction hypothesis

$= Vet_n(E) \vee^b Vet(F \vee^b G)$ Applying Definition 181

$= Vet_n(E) \vee^b (Vet_n(F) \vee^b Vet_n(G))$ Applying Definition 181

This is the result we are seeking

Next I need to show that

$(A \wedge^b B) \wedge^b C = A \wedge^b (B \wedge^b C)$

This is the dual of the previous result.

Next, I need to show:

$(E \vee^b F) \wedge^b F = F$ and $(E \wedge^b F) \vee^b F = F$

Theorem 199 tells us that $E \vee^b F$ is the least upper bound of E and F, in particular $F \leq (E \vee^b F)$. Definition 189 then tells us that $(E \vee^b F) \wedge^b F = F$

The second result is the dual of the first.

Next I need

$(E \vee^b F) \wedge^b G = (E \wedge^b G) \vee^b (F \wedge^b G)$ and $(E \wedge^b F) \vee^b G = (E \vee^b G) \wedge^b (F \vee^b G)$

I will prove these by induction on n.

First $(E \vee^b F) \wedge^b G = (E \wedge^b G) \vee^b (F \wedge^b G)$

In C_0 it is true because there is only one bipartition: $\top_0$.

Let us assume that $E = Dum_n(A)$, $F = Dum_n(B)$ and $G = Dum_n(C)$.

So we need to prove

$(Dum_n(A) \vee^b Dum_n(B)) \wedge^b Dum_n(C) = (Dum_n(A) \wedge^b Dum_n(C)) \vee^b (Dum_n(B) \wedge^b Dum_n(C))$

By Definition 179 and Definition 181 this is true iff

$Dum_n(A \vee^b B) \wedge^b Dum_n(C) = Dum_n(A \wedge^b C) \vee^b Dum_n(B \wedge^b C)$

By the same definitions, this is equivalent to $Dum_n((A \vee^b B) \wedge^b C) = Dum_n((A \wedge^b C) \vee^b (B \wedge^b C))$.

This is true iff (Using Definition 83)

$(A \vee^b B) \wedge^b C = (A \wedge^b C) \vee^b (B \wedge^b C)$.

Which is true by the induction hypothesis.

Let us assume that $E = Dum_n(A)$, $F = Dum_n(B)$ and $G = Vet_n(C)$.

So we need to prove

$(Dum_n(A) \vee^b Dum_n(B)) \wedge^b Vet_n(C) = (Dum_n(A) \wedge^b Vet_n(C)) \vee^b (Dum_n(B) \wedge^b Vet_n(C))$

By Definition 179 and Definition 181 this is true iff

$Dum_n(A \vee^b B) \wedge^b Vet_n(C) = Dum_n(A \wedge^b C) \vee^b Dum_n(B \wedge^b C)$

By the same definitions, this is equivalent to $Dum_n((A \vee^b B) \wedge^b C) = Dum_n((A \wedge^b C) \vee^b (B \wedge^b C))$.

This is true iff (Using Definition 83)

$(A \vee^b B) \wedge^b C = (A \wedge^b C) \vee^b (B \wedge^b C)$.

Which is true by the induction hypothesis.

Let us assume that $E = Dum_n(A)$, $F = Vet_n(B)$ and $G = Dum_n(C)$.

So we need to prove

$(Dum_n(A) \vee^b Vet_n(B)) \wedge^b Dum_n(C) = (Dum_n(A) \wedge^b Dum_n(C)) \vee^b (Vet_n(B) \wedge^b Dum_n(C))$

By Definition 179 and Definition 181 this is true iff

$Vet_n(A \vee^b B) \wedge^b Dum_n(C) = Dum_n(A \wedge^b C) \vee^b Dum_n(B \wedge^b C)$

By the same definitions, this is equivalent to $Dum_n((A \vee^b B) \wedge^b C) = Dum_n((A \wedge^b C) \vee^b (B \wedge^b C))$.

This is true iff (Using Definition 83)

$(A \vee^b B) \wedge^b C = (A \wedge^b C) \vee^b (B \wedge^b C)$.

Which is true by the induction hypothesis.

Let us assume that $E = Dum_n(A)$, $F = Vet_n(B)$ and $G = Vet_n(C)$.

So we need to prove

$(Dum_n(A) \vee^b Vet_n(B)) \wedge^b Vet_n(C) = (Dum_n(A) \wedge^b Vet_n(C)) \vee^b (Vet_n(B) \wedge^b Vet_n(C))$

By Definition 179 and Definition 181 this is true iff

$Vet_n(A \vee^b B) \wedge^b Vet_n(C) = Dum_n(A \wedge^b C) \vee^b Vet_n(B \wedge^b C)$

By the same definitions, this is equivalent to $Vet_n((A \vee^b B) \wedge^b C) = Vet_n((A \wedge^b C) \vee^b (B \wedge^b C))$.

This is true iff (Using Definition 83)

$(A \vee^b B) \wedge^b C = (A \wedge^b C) \vee^b (B \wedge^b C)$.

Which is true by the induction hypothesis.

Let us assume that $E = Vet_n(A)$, $F = Dum_n(B)$ and $G = Dum_n(C)$.

So we need to prove

$(Vet_n(A) \vee^b Dum_n(B)) \wedge^b Dum_n(C) = (Vet_n(A) \wedge^b Dum_n(C)) \vee^b (Dum_n(B) \wedge^b Dum_n(C))$

By Definition 179 and Definition 181 this is true iff

$Vet_n(A \vee^b B) \wedge^b Dum_n(C) = Dum_n(A \wedge^b C) \vee^b Dum_n(B \wedge^b C)$

By the same definitions, this is equivalent to $Dum_n((A \vee^b B) \wedge^b C) = Dum_n((A \wedge^b C) \vee^b (B \wedge^b C))$.

This is true iff (Using Definition 83)

$(A \vee^b B) \wedge^b C = (A \wedge^b C) \vee^b (B \wedge^b C)$.

Which is true by the induction hypothesis.

Let us assume that $E = Vet_n(A)$, $F = Dum_n(B)$ and $G = Vet_n(C)$.

So we need to prove

$(Vet_n(A) \vee^b Dum_n(B)) \wedge^b Vet_n(C) = (Vet_n(A) \wedge^b Vet_n(C)) \vee^b (Dum_n(B) \wedge^b Vet_n(C))$

By Definition 179 and Definition 181 this is true iff

$Vet_n(A \vee^b B) \wedge^b Vet_n(C) = Vet_n(A \wedge^b C) \vee^b Dum_n(B \wedge^b C)$

By the same definitions, this is equivalent to $Vet_n((A \vee^b B) \wedge^b C) = Vet_n((A \wedge^b C) \vee^b (B \wedge^b C))$.

This is true iff (Using Definition 88)

$(A \vee^b B) \wedge^b C = (A \wedge^b C) \vee^b (B \wedge^b C)$.

Which is true by the induction hypothesis.

Let us assume that $E = Vet_n(A)$, $F = Vet_n(B)$ and $G = Dum_n(C)$.

So we need to prove

$(Vet_n(A) \vee^b Vet_n(B)) \wedge^b Dum_n(C) = (Vet_n(A) \wedge^b Dum_n(C)) \vee^b (Vet_n(B) \wedge^b Dum_n(C))$

By Definition 179 and Definition 181 this is true iff

$Vet_n(A \vee^b B) \wedge^b Dum_n(C) = Dum_n(A \wedge^b C) \vee^b Dum_n(B \wedge^b C)$

By the same definitions, this is equivalent to $Dum_n((A \vee^b B) \wedge^b C) = Dum_n((A \wedge^b C) \vee^b (B \wedge^b C))$.

This is true iff (Using Definition 83)

$(A \vee^b B) \wedge^b C = (A \wedge^b C) \vee^b (B \wedge^b C)$.

Which is true by the induction hypothesis.

Let us assume that $E = Vet_n(A)$, $F = Vet_n(B)$ and $G = Vet_n(C)$.

So we need to prove

$(Vet_n(A) \vee^b Vet_n(B)) \wedge^b Vet_n(C) = (Vet_n(A) \wedge^b Vet_n(C)) \vee^b (Vet_n(B) \wedge^b Vet_n(C))$

By Definition 179 and Definition 181 this is true iff

$Vet_n(A \vee^b B) \wedge^b Vet_n(C) = Vet_n(A \wedge^b C) \vee^b Vet_n(B \wedge^b C)$

By the same definitions, this is equivalent to $Vet_n((A \vee^b B) \wedge^b C) = Vet_n((A \wedge^b C) \vee^b (B \wedge^b C))$.

This is true iff (Using Definition 88)

$(A \vee^b B) \wedge^b C = (A \wedge^b C) \vee^b (B \wedge^b C)$.

Which is true by the induction hypothesis.

Next $(E \wedge^b F) \vee^b G = (E \vee^b G) \wedge^b (F \vee^b G)$. This is the dual of the previous result.

Finally, I need to show that, if E is an object of C_n, $E \vee^b E^c = \top_n^c$ and $E \wedge^c E = \top_n$. The second result is the dual of the first. So we only need to prove that $E \vee^b E^c = \top_0^c$.

In C_0 this is true as there is only one bipartition: $\top_0$.

Let us assume that the theorem holds in C_n and let E be a bipartition in C_{n+1}

If $E = Dum_n(A)$ then

$E \vee^b E^c$

$= Dum_n(A) \vee^b Dum_n(A)^c$

$= Dum_n(A) \vee^b Vet_n(A^c)$ By Definition 176

$= Vet_n(A \vee^b A^c)$ By Definition 181

$= Vet_n(\top_n^c)$ By the induction hypothesis

$= Dum_n(\top_n)^c$ By Definition 176

$= \top^c_{n+1}$ By Definition 85

If $E = Vet_n(A)$ then

$E \vee^b E^c$

$= Vet_n(A) \vee^b Vet_n(A)^c$

$= Vet_n(A) \vee^b Dum_n(A^c)$ By Definition 176

$= Vet_n(A \vee^b A^c)$ By Definition 181

$= Vet_n(\top^c_n)$ By the induction hypothesis

$= Dum_n(\top_n)^c$ By Definition 176

$= \top^c_{n+1}$ By Definition 85

$\square$

Comments 203. The operation of taking the dual is a bijection (Theorem 80) of the bipartitions to the inverted bipartitions (Theorem 175). The next few theorems show that this operation respects complement, disjunction, conjunction and partial order. This will allow us to show that the set of inverted bipartitions with its operations and complement is a Boolean Algebra and the operation of taking the dual is an isomorphism.

Definition 204. Every inverted bipartition B is either of the form $Dum_n(D)$ or $Pas_n(D)$ where, D is an inverted bipartition (Theorem 173).

I define *The complement of* B by recursion.

If $B = Dum_n(D)$ then $B^c = Pas_n(D^c)$

If $B = Pas_n(D)$ then $B^c = Dum_n(D^c)$

In C_0, there is only one inverted bipartition:$\perp_0$. I define $\perp^c_0$ to be $\perp_0$.

Theorem 205. *If B is a bipartition in C_n then $(B^c)^* = (B^*)^c$*

Proof. The proof is by induction on n. It is clearly true in C_0 as there is only one inverted bipartition in C_0.

B is a bipartition in C_{k+1} and so it can either be written as $Dum_{k+1}(A)$ or $Vet_{k+1}(A)$. (Theorem 171)

If $B = Dum_{k+1}(A)$ then

$(Dum_{k+1}(A)^c)^*$

$= (Vet_{k+1}(A^c))^*$ Definition 176

$= Pas_{k+1}((A^c)^*)$ Theorem 98

$= Pas_{k+1}((A^*)^c)$ induction hypothesis

$= Dum_{k+1}(A^*)^c$ Definition 204

$= ((Dum_{k+1}(A))^*)^c$ Theorem 101

If $B = Vet_{k+1}(A)$ then

$(Vet_{k+1}(A)^c)^*$

$= Dum_{k+1}(A^c)^*$ Definition 176

$= Dum_{k+1}((A^c)^*)$ Theorem 101

$= Dum_{k+1}((A^*)^c)$ induction hypothesis

$= Pas_{k+1}(A^*)^c$ Definition 204

$= ((Vet_{k+1}(A))^*)^c$ Theorem 98

□

Definition 206. Given inverted bipartitions: A and B, in C_n, the *inverted bipartition conjunction:* $A \wedge^i B$ is defined by recursion on n as follows:

$Pas_n(C) \wedge^i Pas_n(D) = Pas_n(C \wedge^i D)$

$Pas_n(C) \wedge^i Dum_n(D) = Dum_n(C \wedge^i D)$

$Dum_n(C) \wedge^i Pas_n(D) = Dum_n(C \wedge^i D)$

$Dum_n(C) \wedge^i Dum_n(D) = Dum_n(C \wedge^i D)$

In C_0 there is one inverted bipartition: $\bot_0$. $\bot_0 \wedge^i \bot_0 = \bot_0$.

Definition 207. Given inverted bipartitions: A and B, in C_n, the *inverted bipartition disjunction:* $A \vee^i B$ is defined by recursion on n as follows:

$Pas_n(C) \vee^i Pas_n(D) = Pas_n(C \vee^i D)$

$Pas_n(C) \vee^i Dum_n(D) = Pas_n(C \vee^i D)$

$Dum_n(C) \vee^i Pas_n(D) = Pas_n(C \vee^i D)$

$Dum_n(C) \vee^i Dum_n(D) = Dum_n(C \vee^i D)$

In C_0 there is one inverted bipartition: $\bot_0$. $\bot_0 \vee^i \bot_0 = \bot_0$.

Theorem 208. *If A and B are bipartitions in C_n then $(A \wedge^b B)^* = A^* \wedge^i B^*$ and $(A \wedge^i B)^* = A^* \wedge^b B^*$*

Proof. The proof of $(A \wedge^b B)^* = A^* \wedge^i B^*$ is by induction on n.

In C_0 it is true as there is only one inverted bipartition ($\bot_0$). Let us say that we have the result for C_k.

A is a bipartition and so $A = Vet_{k+1}(C)$ or $A = Dum_{k+1}(C)$ where C is a bipartition in C_k. Also B is a bipartition and so $B = Vet_{k+1}(D)$ or $B = Dum_{k+1}(D)$ where D is a bipartition in C_k (Theorem 171).

If $A = Dum_{k+1}(C)$ and $B = Dum_{k+1}(D)$ then

$(A \wedge^b B)^*$

$= (Dum_{k+1}(C) \wedge^b Dum_{k+1}(D))^*$

$= (Dum_{k+1}(C \wedge^b D))^*$ By Definition 179

$= Dum_{k+1}((C \wedge^b D)^*)$ By Theorem 101

$= Dum_{k+1}(C^* \wedge^i D^*)$ By the induction hypothesis

$= Dum_{k+1}(C^*) \wedge^i Dum_{k+1}(D^*)$ By Definition 206

$= Dum_{k+1}(C)^* \wedge^i Dum_{k+1}(D)^*$ By Theorem 101

$A^* \wedge^i B^*$

If $A = Vet_{k+1}(C)$ and $B = Dum_{k+1}(D)$ then

$(A \wedge^b B)^*$

$= (Vet_{k+1}(C) \wedge^b Dum_{k+1}(D))^*$

$= (Dum_{k+1}(C \wedge^b D))^*$ By Definition 179

$= Dum_{k+1}((C \wedge^b D)^*)$ By Theorem 101

$= Dum_{k+1}(C^* \wedge^i D^*)$ By the induction hypothesis

$= Vet_{k+1}(C^*) \wedge^i Dum_{k+1}(D^*)$ By Definition 206

$= Vet_{k+1}(C)^* \wedge^i Dum_{k+1}(D)^*$ By Theorem 101

$A^* \wedge^i B^*$

If $A = Dum_{k+1}(C)$ and $B = Vet_{k+1}(D)$ then

$(A \wedge^b B)^*$

$= (Dum_{k+1}(C) \wedge^b Vet_{k+1}(D))^*$

$= (Dum_{k+1}(C \wedge^b D))^*$ By Definition 179

$= Dum_{k+1}((C \wedge^b D)^*)$ By Theorem 101

$= Dum_{k+1}(C^* \wedge^i D^*)$ By the induction hypothesis

$= Dum_{k+1}(C^*) \wedge^i Vet_{k+1}(D^*)$ By Definition 206

$= Dum_{k+1}(C)^* \wedge^i Vet_{k+1}(D)^*$ By Theorem 101

$A^* \wedge^i B^*$

If $A = Vet_{k+1}(C)$ and $B = Vet_{k+1}(D)$ then

$(A \wedge^b B)^*$

$= (Vet_{k+1}(C) \wedge^b Vet_{k+1}(D))^*$

$= (Vet_{k+1}(C \wedge^b D))^*$ By Definition 179

$= Vet_{k+1}((C \wedge^b D)^*)$ By Theorem 101

$= Vet_{k+1}(C^* \wedge^i D^*)$ By the induction hypothesis

$= Vet_{k+1}(C^*) \wedge^i Vet_{k+1}(D^*)$ By Definition 206

$= Vet_{k+1}(C)^* \wedge^i Vet_{k+1}(D)^*$ By Theorem 101

$A^* \wedge^i B^*$

Now we have completed the proof that $(A \wedge^b B)^* = A^* \wedge^i B^*$.

Taking the dual of both sides gives $((A \wedge^b B)^*)^* = (A^* \wedge^i B^*)^*$.

Theorem 75 tells us that

$A \wedge^b B = (A^* \wedge^i B^*)^*$.

Now, let us substitute $C^* = A$ and $D^* = B$ (We can do this because * is self-inverse and hence onto - Theorem 75)

$C^* \wedge^b D^* = ((C^*)^* \wedge^i (D^*)^*)^* \iff$ Using Theorem 75

$C^* \wedge^b D^* = (C \wedge^i D)^*$ This is the other result that we were trying to establish.

□

Theorem 209. *If A and B are bipartitions in C_n then $(A \vee^b B)^* = A^* \vee^i B^*$ and $(A \vee^i B)^* = A^* \vee^b B^*$*

Proof. The proof of $(A \vee^b B)^* = A^* \vee^i B^*$ is by induction on n.

In C_0 it is true as there is only one inverted bipartition $(\perp_0)$. Let us say that we have the result for C_k.

A is a bipartition and so $A = Vet_{k+1}(C)$ or $A = Dum_{k+1}(C)$ where C is a bipartition in C_k. Also B is a bipartition and so $B = Vet_{k+1}(D)$ or $B = Dum_{k+1}(D)$ where D is a bipartition in C_k (Theorem 171).

If $A = Dum_{k+1}(C)$ and $B = Dum_{k+1}(D)$ then

$(A \vee^b B)^*$

$= (Dum_{k+1}(C) \vee^b Dum_{k+1}(D))^*$

$= (Dum_{k+1}(C \vee^b D))^*$ By Definition 181

$= Dum_{k+1}((C \vee^b D)^*)$ By Theorem 101

$= Dum_{k+1}(C^* \vee^i D^*)$ By the induction hypothesis

$= Dum_{k+1}(C^*) \vee^i Dum_{k+1}(D^*)$ By Definition 207

$= Dum_{k+1}(C)^* \vee^i Dum_{k+1}(D)^*$ By Theorem 101

$A^* \vee^i B^*$

If $A = Vet_{k+1}(C)$ and $B = Dum_{k+1}(D)$ then

$(A \vee^b B)^*$

$= (Vet_{k+1}(C) \vee^b Dum_{k+1}(D))^*$

$= (Vet_{k+1}(C \vee^b D))^*$ By Definition 181

$= Vet_{k+1}((C \vee^b D)^*)$ By Theorem 101

$= Vet_{k+1}(C^* \vee^i D^*)$ By the induction hypothesis

$= Vet_{k+1}(C^*) \vee^i Dum_{k+1}(D^*)$ By Definition 207

$= Vet_{k+1}(C)^* \vee^i Dum_{k+1}(D)^*$ By Theorem 101

$A^* \vee^i B^*$

If $A = Dum_{k+1}(C)$ and $B = Vet_{k+1}(D)$ then

$(A \vee^b B)^*$

$= (Dum_{k+1}(C) \vee^b Vet_{k+1}(D))^*$

$= (Vet_{k+1}(C \vee^b D))^*$ By Definition 181

$= Vet_{k+1}((C \vee^b D)^*)$ By Theorem 101

$= Vet_{k+1}(C^* \vee^i D^*)$ By the induction hypothesis

$= Dum_{k+1}(C^*) \vee^i Vet_{k+1}(D^*)$ By Definition 207

$= Dum_{k+1}(C)^* \vee^i Vet_{k+1}(D)^*$ By Theorem 101

$A^* \vee^i B^*$

If $A = Vet_{k+1}(C)$ and $B = Vet_{k+1}(D)$ then

$(A \vee^b B)^*$

$= (Vet_{k+1}(C) \vee^b Vet_{k+1}(D))^*$

$= (\mathit{Vet}_{k+1}(C \vee^b D))^*$ By Definition 181

$= \mathit{Vet}_{k+1}((C \vee^b D)^*)$ By Theorem 101

$= \mathit{Vet}_{k+1}(C^* \vee^i D^*)$ By the induction hypothesis

$= \mathit{Vet}_{k+1}(C^*) \vee^i \mathit{Vet}_{k+1}(D^*)$ By Definition 207

$= \mathit{Vet}_{k+1}(C)^* \vee^i \mathit{Vet}_{k+1}(D)^*$ By Theorem 101

$A^* \vee^i B^*$

Now we have completed the proof that $(A \vee^b B)^* = A^* \vee^i B^*$.

Taking the dual of both sides gives $((A \vee^b B)^*)^* = (A^* \vee^i B^*)^*$.

Theorem 75 tells us that

$A \vee^b B = (A^* \vee^i B^*)^*$.

Now, let us substitute $C^* = A$ and $D^* = B$ (We can do this because * is self-inverse and hence onto - Theorem 75)

$C^* \vee^b D^* = ((C^*)^* \vee^i (D^*)^*)^* \iff$ Using Theorem 75

$C^* \vee^b D^* = (C \vee^i D)^*$ This is the other result that we were trying to establish.

□

Comments 210. These Theorems now allow us to establish Theorem 178 to Theorem 202 for inverted bipartitions. Let's take a look.

Theorem 211. *If B is an inverted bipartition then $(B^c)^c = B$*

Proof. $(B^c)^c = B$ Dual is a bijection, Theorem 80

$\iff ((B^c)^c)^* = B^*$ Taking the Dual of both sides

$\iff ((B^c)^*)^c = B^*$ Theorem 205

$\iff ((B^*)^c)^c = B^*$ Theorem 205

This is true by Theorem 178

□

Comments 212. The rest of the proofs are similar

Theorem 213. *If B and D are inverted bipartitions then $(B \wedge^i D)^c = B^c \vee^i D^c$ and $(B \vee^i D)^c = B^c \wedge^i D^c$*

Proof. As in the last example, just apply the isomorphism (take the dual) to either side. This then falls through the various operations. The result is then true by Theorem 187.

$\square$

Theorem 214. $A \wedge^i B = A$ *iff* $A \vee^i B = B$

Proof. This is true by Theorem 188 and the fact that * is an isomorphism between bipartitions and inverted bipartitions.

$\square$

Definition 215. By Theorem 214 $A \wedge^i B = A$ iff $A \vee^i B = B$. In both cases, we say that $A \leq^i B$.

Theorem 216. *Every inverted bipartition is of the form $Dum_n(C)$ or $Pas_n(C)$ with C an inverted bipartition (Theorem 173). If $A \leq^i B$ and A is of the form $Pas_n(C)$ then B is of the form $Pas_n(D)$ where D is also an inverted bipartition.*

Proof. Let us say that $Pas_n(C) \leq^i Dum_n(D)$. Definition 215 tells us that $Pas_n(C) \vee^i Dum_n(D) = Dum_n(D)$. Definition 207 then gives us $Pas_n(C \vee^i D) = Dum_n(D)$. Definition 90, Definition 83 and Definition 94 now tell us that $Cod_{n+1}(Pas_n(C \vee^b D)) = \top_k$ and $Cod_{n+1}(Dum_n(D)) = D$. But $\top_k$ is not an inverted bipartition (Definition 147). This contradiction shows us that $Pas_n(C \vee^i D) \leq^b Dum_n(D)$ was not possible.

$\square$

Theorem 217. *$Dum_n(A) \leq^i Dum_n(B) \iff A \leq^i B$*

$Dum_n(A) \leq^i Pas_n(B) \iff A \leq^i B$

$Pas_n(A) \leq^i Pas_n(B) \iff A \leq^i B$

Of course these are the only possibilities (Theorem 173 and Theorem 216)

Proof. This is clear from Theorem 192 and the fact that * is an isomorphism.

$\square$

Theorem 218. *$A \leq^b B \iff A^* \leq^i B^*$*

Proof. $A \leq^b B$

$\iff A \wedge^b B = A$ Using Definition 189

$\iff (A \wedge^b B)^* = A^*$ taking duals

$\iff A^* \wedge^i B^* = A^*$ Using Theorem 208

$\iff A^* \leq^i B^*$ Using Definition 215

$\square$

Theorem 219. *$\leq^i$ is reflexive on the set of all inverted bipartitions.*

Proof. This is clear from Theorem 193 and the fact that * is an isomorphism.

$\square$

Theorem 220. *$A \leq^i B$ and $B \leq^i A$ imply $A = B$ on the set of all inverted bipartitions.*

Proof. This is clear from Theorem 194 and the fact that * is an isomorphism.

$\square$

Theorem 221. *$\leq^i$ is transitive on C_n. That is $A \leq^i B$ and $B \leq^i C$ imply $A \leq^i C$*

Proof. This is clear from Theorem 195 and the fact that * is an isomorphism. $\square$

Theorem 222. *$\leq^i$ is a partial order on the set of all invereted bipartitions in C_n.*

Proof. $\leq^i$ is reflexive (Theorem 219), antisymmetric (Theorem 220) and transitive (Theorem 221). $\square$

Theorem 223. *If A is an inverted bipartition in C_n then $Dum_n(A) \leq^i Pas_n(A)$*

Proof. This is true by Theorem 197 and the fact that * is an isomorphism. $\square$

Theorem 224. $E \leq^i F \iff F^c \leq^i E^c$

Proof. $E \leq^i F$

$\iff E = E \wedge^i F$ Using Definition 215

$\iff E^c = E^c \vee^i F^c$ Taking complements and using Theorem 213

$\iff F^c \leq^i E^c$ Using Definition 215

$\square$

Theorem 225. *$A \vee^i B$ is the least upper bound of A and B with respect to $\leq^i$*

Proof. This can be deduced from Theorem 199 $\square$

Theorem 226. *$A \wedge^i B$ is the greatest lower bound of A and B with respect to $\leq^i$*

Proof. This can be deduced from Theorem 200

□

Theorem 227. *For all inverted bipartitions A in C_n, $A \wedge^i \perp_n = \perp_n$*

Proof. $A \wedge^i \perp_n = \perp_n$

$\iff$ $A^* \wedge^b \top_n = \top_n$ Taking the dual of both side and using Theorem 208 and Theorem 87

Which is true by Theorem 183

□

Theorem 228. *For all inverted bipartitions A in C_n, $A \vee^i \perp_n = A$*

Proof. $A \vee^i \perp_n = A$

$\iff$ $A^* \vee^b \top_n = A^*$ Taking the dual of both side and using Theorem 209 and Theorem 87

Which is true by Theorem 185

□

Theorem 229. *$\perp_n$ is the minimal inverted bipartition in C_n relative to $\leq^i$*

Proof. For all inverted bipartitions A, $A \vee^i \perp_n = A$ by Theorem 185 and so $\perp_n \leq^i A$ (By Definition 215)

□

Theorem 230. *The set of inverted bipartitions in C_n with two binary operations: invereted bipartition conjunction ($\wedge^i$) and inverted bipartition disjunction ($\vee^i$) and two defined objects $\perp_n$ (as the 0 in the BA) and its complement: $\perp_n^c$ (as the 1 in the BA). constitute a Boolean Algebra*

Proof. Taking the dual is an isomorphism of the Boolean Algebra in Theorem 202 onto the set of all inverted bipartitions with the operations and individuals described.

□

Comments 231. Of course, we have two operations on the whole of C_n: $\wedge$ and $\vee$. These operations satisfy some of the axioms of a Boolean Algebra. Could the whole of C_n be a Boolean Algebra with duality playing the role of complement? This is not possible because the number of objects in C_n is not a power of 2 ([6, Theorem 4.5]. Which of the axioms are satisfied? How close are we?

Theorem 232. *C_n is a distributive lattice but is not complemented.*

Proof. $E \wedge F = F \wedge E$ products in any category are commutative.

$E \vee F = F \vee E$ coproducts in any category are commutative.

$E \wedge (F \wedge G) = (E \wedge F) \wedge G$ products in any category are associative.

$E \vee (F \vee G) = (E \vee F) \vee G$ coproducts in any category are associative.

Next, I need to show that $(E \vee F) \wedge F = F$ and $(E \wedge F) \vee F = F$.

Theorem 134 tells us that $F \leq (E \vee F)$. Theorem 135 then gives $(E \vee F) \wedge F = F$

Theorem 135 tells us that $(E \wedge F) \leq F$. Theorem 135 then gives $(E \wedge F) \vee F = F$

Next I need

$(E \vee F) \wedge G = (E \wedge G) \vee (F \wedge G)$ and $(E \wedge F) \vee G = (E \vee G) \wedge (F \vee G)$.

These are given directly by Theorem 132 and Theorem 133 respectively.

The last step would be $E \wedge E^* = \perp_n$ and $E \vee E^* = \top_n$. This is not true in general. For example $Dict_n^* = Dict_n$ (Theorem 138) and $Dict_n \wedge Dict_n =$

$Dict_n$ and $Dict_n \vee Dict_n = Dict_n$. In fact, it is only true if $E = \bot_n$ or $E = \top_n$.

$\square$

Theorem 233. *There are* 2^n *bipartitions and* 2^n *inverted bipartitions in* C_n

Proof. The fact that the bipartitions are the objects of a finite Boolean Algebra tells us that there are 2^m of them for some m.

The proof that there are 2^n bipartitions in C_n is a simple induction. In C_0, there is $1 = 2^0$ bipartitions ($\top_0$) and $1 = 2^0$ inverted bipartitions: $\bot_0$.

Let us assume that the theorem holds for $n = k$. In C_{k+1}, the bipartitions are of the form $Vet_{k+1}(A)$ or $Dum_{k+1}(A)$ where A is a bipartition of C_k. The inverted bipartitions are of the form $Pas_{k+1}(A)$ or $Dum_{k+1}(A)$ where A is an inverted bipartition. $Vet_{k+1}(A) \neq Dum_{k+1}(A)$ and $Pas_{k+1}(A) \neq Dum_{k+1}(A)$. By Definitions 88, 83 and 90, $\bot_0$ is not a bipartition and $\top_0$ is not an inverted bipartition (Definitions 145 and 147). So there are twice as many bipartitions in C_{k+1} as there are in C_k. There are also twice as many inverted bipartitions in C_{k+1} as there are in C_k. By the induction hypothesis, there are 2^k of each in C_k and so there are 2^{k+1} of each in C_{k+1}.

$\square$

4.4 When Does a Bipartition Win a Game?

Definition 234. All objects, G, of C_n can be considered as games.

I say that a bipartition (B) *wins a game* G iff there is an arrow from B to G.

Definition 235. All objects, G, of C_n can be considered as games.

I say that an inverted bipartition (B) *wins a game* G iff there is an arrow from G^* to B.

Theorem 236. *A bipartition (B) wins a game G iff the inverted bipartition B^* (It is an inverted bipartition by Theorem 175) also wins G*

Proof. B wins G

$\iff$ There is an arrow from B to G Using Definition 234

$\iff$ There is an arrow from G^* to B^* Using Theorem 77 and Definition 66

$\iff$ B^* wins G. Using Definition 235.

□

Comments 237. Before we explore the relationship between games and the bipartitions and inverted bipartitions that win them, we need to relate the operations on, and relations between, bipartitions and inverted bipartitions to the operations on and relations between objects in the category.

Theorem 238. *If E and F are bipartitions in C_n then $E \vee^b F = E \wedge F$*

Proof. The proof is by induction on n.

There is only one bipartition in C_0:$\top_0$. $\top_0 \vee^b \top_0 = \top_0$ and $\top_0 \wedge \top_0 = \top_0$

Let us assume that we have the result in C_k. Let E and F be bipartitions of C_{k+1}. E and F must be of the form $Dum_{k+1}(A)$ or $Vet_{k+1}(A)$ where A is a bipartition of C_k (Theorem 171)

First, let us assume that $E = Dum_{k+1}(A)$ and $F = Dum_{k+1}(B)$ are bipartitions

$E \vee^b F$

$= Dum_{k+1}(A) \vee^b Dum_{k+1}(B)$ By Definition

$= Dum_{k+1}(A \vee^b B)$ Using Definition 181

$= Dum_{k+1}(A \wedge B)$ By the induction hypothesis

$= Dum(A) \wedge Dum(B)$ By Definition 83 and Theorem 120

$E \wedge F$

Let us assume that $E = Vet(A)$ and $F = Dum(B)$ are bipartitions

$E \vee^b F$

$= Vet(A) \vee^b Dum(B)$ By Definition

$= Vet(A \vee^b B)$ Using Definition 181

$= Vet(A \wedge B)$ By the induction hypothesis

$= Vet(A) \wedge Dum(B)$ By Definition 83, Definition 88, Theorem 112, Theorem 104 and Theorem 120

$E \wedge F$

Let us assume that $E = Dum(A)$ and $F = Vet(B)$ are bipartitions

$E \vee^b F$

$= Dum(A) \vee^b Vet(B)$ By Definition

$= Vet(A \vee^b B)$ Using Definition 181

$= Vet(A \wedge B)$ By the induction hypothesis

$= Dum(A) \wedge Vet(B)$ By Definition 83, Definition 88, Theorem 112, Theorem 104 and Theorem 120

$E \wedge F$

Let us assume that $E = Vet(A)$ and $F = Vet(B)$ are bipartitions

$E \vee^b F$

$= Vet(A) \vee^b Vet(B)$ By definition

$= Vet(A \vee^b B)$ Using Definition 181

$= Vet(A \wedge B)$ By the induction hypothesis

$= Vet(A) \wedge Vet(B)$ By Definition 88, Theorem 112, Theorem 104 and Theorem 120

$E \wedge F$

$\square$

Theorem 239. *If E and F are bipartitions then $E \wedge F = (E^c \wedge^b F^c)^c$*

Proof. There is only one bipartition in C_0:$\top_0$. $\top_0 \wedge \top_0 = \top_0$ and $(\top_0^c \wedge^b \top_0^c)^c = (\top_0 \wedge^b \top_0)^c = \top_0^c = \top_0$

Let us assume that we have the result in C_k. Let E and F be bipartitions of C_{k+1}. They can both be of the form $Vet_{k+1}(A)$ and $Dum_{k+1}(A)$ (By Theorem 171)

First let us assume $E = Dum_{k+1}(A)$ and $F = Dum_{k+1}(B)$.

$(E^c \wedge^b F^c)^c$

$= (Dum_{k+1}(A)^c \wedge^b Dum_{k+1}(B)^c))^c$ By Definition

$= (Vet_{k+1}(A^c) \wedge^b Vet_{k+1}(B^c))^c$ Using Definition 176.

$= (Vet_{k+1}((A^c) \wedge^b (B^c)))^c$ Using Definition 179

$= Dum_{k+1}(((A^c) \wedge^b (B^c))^c)$ Using Definition 176

$= Dum_{k+1}(A \wedge B)$ By the induction hypothesis

$= Dum_{k+1}(A) \wedge Dum_{k+1}(B)$ By Definition 83 and Theorem 120

$= E \wedge F$ By definition

Let us assume $E = Vet_{k+1}(A)$ and $F = Dum_{k+1}(B)$.

$(E^c \wedge^b F^c)^c$

$= (Vet_{k+1}(A)^c \wedge^b Dum_{k+1}(B)^c))^c$ By Definition

$= (Dum_{k+1}(A^c) \wedge^b Vet_{k+1}(B^c))^c$ Using Definition 176.

$= (Dum_{k+1}((A^c) \wedge^b (B^c)))^c$ Using Definition 179

$= Vet_{k+1}(((A^c) \wedge^b (B^c))^c)$ Using Definition 176

$= Vet_{k+1}(A \wedge B)$ By the induction hypothesis

$= Vet_{k+1}(A) \wedge Dum_{k+1}(B)$ By Definition 83, Definition 88 Theorem 104, Theorem 112 and Theorem 120

$= E \wedge F$ By definition

Let us assume $E = Dum_{k+1}(A)$ and $F = Vet_{k+1}(B)$.

$(E^c \wedge^b F^c)^c$

$= (Dum_{k+1}(A)^c \wedge^b Vet_{k+1}(B)^c))^c$ By Definition

$= (Vet_{k+1}(A^c) \wedge^b Dum_{k+1}(B^c))^c$ Using Definition 176.

$= (Dum_{k+1}((A^c) \wedge^b (B^c)))^c$ Using Definition 179

$= Vet_{k+1}(((A^c) \wedge^b (B^c))^c)$ Using Definition 176

$= Vet_{k+1}(A \wedge B)$ By the induction hypothesis

$= Dum_{k+1}(A) \wedge Vet_{k+1}(B)$ By Definition 83, Definition 88 Theorem 104, Theorem 112 and Theorem 120

$= E \wedge F$ By definition

Let us assume $E = Vet_{k+1}(A)$ and $F = Vet_{k+1}(B)$.

$(E^c \wedge^b F^c)^c$

$= (Vet_{k+1}(A)^c \wedge^b Vet_{k+1}(B)^c))^c$ By Definition

$= (Dum_{k+1}(A^c) \wedge^b Dum_{k+1}(B^c))^c$ Using Definition 176.

$= (Dum_{k+1}((A^c) \wedge^b (B^c)))^c$ Using Definition 179

$= Vet_{k+1}(((A^c) \wedge^b (B^c))^c)$ Using Definition 176

$= Vet_{k+1}(A \wedge B)$ By the induction hypothesis

$= Vet_{k+1}(A) \wedge Vet_{k+1}(B)$ By Definition 88 Theorem 104, Theorem 112 and Theorem 120

$= E \wedge F$ By definition

This covers all the cases and completes the induction.

□

Theorem 240. *If E and F are bipartitions then $E \leq^b F \iff F \leq E$*

Proof. $E \leq^b F$

$\iff F = E \vee^b F$ Using Definition 189

$\iff F = E \wedge F$ Using Theorem 238

$\iff F \leq E$ Using Theorem 135

□

Theorem 241. *If E and F are inverted bipartitions then $E \vee^i F = E \vee F$*

Proof. $E \vee^i F = E \vee F$

$\iff E^* \vee^b F^* = E^* \wedge F^*$ Taking duals and using Theorem 209 and Theorem 116

Which is true by Theorem 238

□

Theorem 242. *If E and F are inverted bipartitions then $E \wedge^i F = (E^c \vee F^c)^c$*

Proof. $E \wedge^i F = (E^c \vee F^c)^c$

$\iff E^* \wedge^b F^* = ((E^*)^c \wedge (F^*)^c)^c$ Taking the dual and the using Theorem 205, Theorem 208 and Theorem 116.

Which is true by Theorem 239.

□

Theorem 243. *If E and F are inverted bipartitions $(E \leq^i F) \iff (E \leq F)$*

Proof. $E \leq^i F$

$\iff F = E \vee^i F$ Using Definition 215

$\iff F = E \vee F$ Using Theorem 241

$\iff E \leq F$ Using Definition 134

□

Theorem 244. *If a bipartition, B wins G and $B \leq^b C$ then C wins G.*

Proof. If B wins G there is an arrow from B to G (Definition 234).

$B \leq^b C$

$\implies C \leq B$ Using Theorem 240.

$\implies$ that there is an arrow from C to B (Using Definition 66). Combine this with the arrow from B to G and this produces an arrow from C to G. This shows that C wins G (Definition 234).

□

Comments 245. This says that all of the objects of C_n are monotonic (relative to the notion of a bipartition winning a game).

Theorem 246. *If an inverted bipartition, B wins G and $B \leq^i C$ then C wins G.*

Proof. This is the dual of Theorem 244.

□

Comments 247. This says that all of the objects of C_n are monotonic.

Theorem 248. *If a bipartition B wins G and $G \leq H$ then B wins H.*

Proof. If B wins G then there is an arrow from B to G (Definition 234).

If $G \leq H$ then there is an arrow from G to H (Definition 66).

Combining these gives an arrow from B to H showing that B wins H (Definition 234).

□

Comments 249. This shows us that $\leq$ aligns with the idea of a bigger game.

Theorem 250. *If A is a bipartition and G is a member of C_n then (A wins $Dom_n(G)$) iff ($Dum_n(A)$ wins G) implies ($Vet_n(A)$ wins G)*

Proof. A wins $Dom_n(G)$

$\iff$ There is an arrow from A to $Dom_n(G)$ (Definition 234)

$\iff$ There are arrows from A to $Dom_n(G)$ and $Cod_n(G)$ (There is always an arrow from $Dom_n(G)$ to $Cod_n(G)$ (Theorem 68 and Definition 66). Combine the arrows.)

$\iff$ There is an arrow from $Dum_n(A)$ to G (Definition 83)

$Dum_n(A)$ wins G

$\implies$ There is an arrow from $Dum_n(A)$ to G (Definition 234)

$\implies$ There is an arrow from A to $Cod(G)$

$\implies$ There is an arrow from A to $Cod(G)$ and an arrow from $\perp_n$ to $Dom(G)$ (Theorem 102)

$\implies$ There is an arrow from $Vet(A)$ to G (Definition 88)

$\implies$ $Vet(A)$ wins G (Definition 234)

□

Theorem 251. *If A is a bipartition and G is a member of C_n then (A wins $Dom_n(G)$) implies (A wins $Cod_n(G)$) iff ($Vet_n(A)$ wins G)*

Proof. A wins $Dom_n(G)$

$\implies$ There is an arrow from A to $Dom_n(G)$ (Definition 234)

$\implies$ There is an arrow from A to $Cod_n(G)$ (There is always an arrow from $Dom_n(G)$ to $Cod_n(G)$ (Theorem 68 and Definition 66). Combine the arrows.)

$\implies$ A wins $Cod_n(G)$ Definition 234

$\iff$ There is an arrow from A to $Cod_n(G)$ (Definition 234)

$\iff$ There are arrows from A to $Cod_n(G)$ and from $\perp_n$ to $Dom_n(G)$ (There is always an arrow from $\perp_n$ to $Dom_n(G)$ (Theorem 102))

$\iff$ There is an arrow from $Vet_n(A)$ to G (Definition 88)

$\iff$ $Vet_n(A)$ wins G (Definition 234)

□

Theorem 252. *If A is an inverted bipartition and G is a member of C_n then (A wins $Dom_n(G)$) iff ($Dum_n(A)$ wins G) implies ($Pas_n(A)$ wins G)*

Proof. This is the dual of Theorem 250

□

Theorem 253. *If A is an inverted bipartition and G is a member of C_n then (A wins $Dom_n(G)$) implies (A wins $Cod_n(G)$) iff ($Pas_n(A)$ wins G)*

Proof. This is the dual of Theorem 251

□

Theorem 254. *If (for all bipartitions B, B wins $G \implies B$ wins H) then $H \geq G$.*

Proof. The proof is by induction. It is clearly true in C_0.

Let us say we have the result in C_k.

Let us also say that, in C_{k+1}, every B that wins G also wins H. We need to show that $H \geq G$.

I need to show that every B that wins $Dom(G)$ also wins $Dom(H)$ and every B that wins $Cod(G)$ also wins $Cod(H)$. The Induction Hypothesis will then tell us that $Dom(H) \geq Dom(G)$ and $Cod(H) \geq Cod(G)$. From Theorem 70 we can then complete the induction and deduce that $H \geq G$.

Let us assume that B wins $Dom(G)$. Then (by Theorem 250) we know that $Dum_{k+1}(B)$ wins G. By hypothesis, we can infer that $Dum_{k+1}(B)$ wins H. Theorem 250 then tells us that B wins $Dom(H)$.

Let us assume that B wins $Cod(G)$. Then (by Theorem 251) we know that $Vet_{k+1}(B)$ wins G. By hypothesis, we can infer that $Vet_{k+1}(B)$ wins H. Theorem 251 then tells us that B wins $Cod(H)$.

□

Theorem 255. *If an inverted bipartition B wins G and $G \leq H$ then B wins H.*

Proof. This is the dual of Theorem 248.

□

Comments 256. This shows us that $\leq$ aligns with the idea of a bigger game.

Theorem 257. *If (for all inverted bipartitions, I, I wins $G \implies I$ wins H) then $H \geq G$.*

Proof. This is the dual of Theorem 254

□

Theorem 258. *A bipartition B wins $F \wedge G$ iff it wins F and G.*

Proof. Let us say that B wins $F \wedge G$. Then there is an arrow from $f : B \to F \wedge G$ (Definition 234). There are projection maps from $F \wedge G$ to F and G. Combining these with f gives arrows from B to F and G and so B wins F and B wins G (Definition 234).

Let us say that B wins F and G. There are arrows from B to F and B to G (Definition 234). $F \wedge G$ is a product and so there is an arrow from B to $F \wedge G$. This means that B wins $F \wedge G$ (Definition 234).

$\square$

Theorem 259. *An inverted bipartition B wins $F \wedge G$ iff it wins F and G.*

Proof. This is the dual of Theorem 258.

$\square$

Theorem 260. *A bipartition B in C_n wins G^* iff B^c doesn't win G.*

Proof. The proof is by induction on n.

In C_0 there is only one bipartition: $\top_0$.

There are two games: $\top_0$ and $\perp_0$.

$\top_0$ does not win $(\top_0)^* = \perp_0$ but $(\top_0)^c = \top_0$ does win $\top_0$

$\top_0$ does win $(\perp_0)^* = \top_0$ but $(\top_0)^c = \top_0$ does not win $\perp_0$

Let us say that the theorem holds in C_k.

Let B be a bipartition in C_{k+1} and G be a game in C_{k+1}. By Theorem 171, $B = Dum_{k+1}(A)$ or $Vet_{k+1}(A)$ with A a bipartition.

First, I will assume that $B = Dum_{k+1}(A)$.

B wins G^*

$\iff$ There is an arrow from B to G^* (Using Definition 234)

$\iff$ There is an arrow from $Dum_{k+1}(A)$ to G^*

$\iff$ There are arrows from A to $Dom_{k+1}(G^*)$ and $Cod_{k+1}(G^*)$ (Using Definition 83, Definition 92 and Definition 94.)

$\iff$ There are arrows from A to $Cod_{k+1}(G)^*$ and $Dom_{k+1}(G)^*$ (Using Theorem 72)

$\iff$ A wins $Cod_{k+1}(G)^*$ and A wins $Dom_{k+1}(G)^*$ (Using Definition 234)

$\iff$ A^c loses $Cod_{k+1}(G)$ and A^c loses $Dom_{k+1}(G)$ Using the induction hypothesis.

$\iff$ There is no arrow from A^c to $Cod_{k+1}(G)$ or from A^c to $Dom_{k+1}(G)$ (Using Definition 234.)

$\iff$ There is no arrow from $[\perp_k, A^c]$ to G (The lack of an arrow from A^c to $Cod_{k+1}(G)$ is all that is required for $\implies$. Going in the other direction ($\impliedby$), the lack of an arrow from A^c to $Cod_{k+1}(G)$ means that there must be no arrow from A^c to $Dom_{k+1}(G)$ because the combination of that with the arrow from $Dom_{k+1}(G)$ to $Cod_{k+1}(G)$ (there always is one by Theorem 68) and Definition 66 would give an arrow from A^c to $Cod_{k+1}(G)$). To go in this direction, we also need that fact that $\perp_k$ is initial.

$\iff$ There is no arrow from $Dum(A)^c$ to G (Using Definition 176)

$\iff$ There is no arrow form B^c to G

$\iff$ B^c loses G Using Definition 234

Next, I will assume that $B = Vet_{k+1}(A)$.

B wins G^*

$\iff$ There is an arrow from B to G^* Using Definition 234

$\iff$ There is an arrow from $Vet(A)$ to G^*

$\iff$ There is an arrow from A to $Cod(G^*)$ (Using Definition 88, Definition 94 and the fact that $\perp_k$ is initial.)

$\iff$ There is an arrow from A to $Dom(G)^*$ (Using Theorem 72)

$\iff$ A wins $Dom(G)^*$ Using Definition 234

$\iff$ A^c loses $Dom(G)$ Using the induction hypothesis.

$\iff$ There is no arrow from A^c to $Dom(G)$ Using Definition 234.

$\iff$ There is no arrow from $Dum(A^c)$ to G (Using Definition 83, Theorem 68 and Theorem 70)

$\iff$ There is no arrow from $Vet(A)^c$ to G (Using Definition 176)

$\iff$ There is no arrow from B^c to G

$\iff$ B^c loses G. (Using Definition 234)

□

Comments 261. Of course, this tells us that the dual behaves as it should.

Theorem 262. *An inverted bipartition B wins G^* iff B^c doesn't win G.*

Proof. This is the dual of Theorem 260

□

Theorem 263. *A bipartition B wins $F \vee G$ iff it wins F or G.*

Proof. B wins $F \vee G$

$\iff$ B^c loses $(F \vee G)^*$ Using Theorem 260

$\iff$ B^c loses $F^* \wedge G^*$ By Theorem 116

$\iff$ B^c loses F^* or B^c loses G^* By Theorem 258

$\iff$ B wins F or B wins G Using Theorem 260

□

Theorem 264. *An inverted bipartition B wins $F \vee G$ iff it wins F or G.*

Proof. This is the dual of Theorem 264

□

4.5 Minimum Winning Bipartitions

Definition 265. If a bipartition, B wins G and $C \leq^b B$ and C wins G implies $B = C$, we say that B is a minimal winning bipartition.

Theorem 266. *For every game G and winning bipartition B, there is a minimal winning bipartition of C_n that is $\leq^b B$*

Proof. Let us say that B is not a MWB. By Definition 265 there is a winning bipartition $B' \leq^b B$ with $B' \neq B$. Is this a MWB? If it is then we are done. If it is not then we just find another winning bipartition $B'' \leq^b B'$ with $B'' \neq B'$. There are a finite number of objects in C_n and so this process must terminate. The only way it can terminate is with B^r a MWB for some r.

□

Definition 267. If an inverted bipartition, B wins G and $C \leq^i B$ and C wins G implies $B = C$, we say that B is a minimal winning inverted bipartition.

Theorem 268. *For every game G and winning inverted bipartition B, there is a minimal winning inverted bipartition of G that is $\leq^i B$*

Proof. This is the dual of Theorem 266.

□

Theorem 269. *Taking the dual maps minimal winning bipartitions of G to minimal winning inverted bipartitions of G and minimal winning inverted bipartitions to minimal winning bipartitions. This mapping is bijective.*

Proof. We know that the dual operation maps bipartitions to inverted bipartitions and vica versa. (By Theorem 175)

We also know that, B wins for G iff B^* wins for G (Theorem 236).

Let us say that B is a minimal winning bipartition for G. I need to show that B^* is a minimal winning inverted bipartition for G.

B^* wins G. Let us assume that $A \leq^i B^*$ and A wins G. We know that A^* wins G (Theorem 236) and $A^* \leq^b B$ (Theorem 218 and Theorem 75) and so $A^* = B$ (Definition 265) and so $A = B^*$ (taking the dual of both sides and using Theorem 75) and B^* is a minimal winning inverted bipartition of G.

Let us say that B is a minimal winning inverted bipartition for G. I need to show that B^* is a minimal winning inverted bipartition for G.

B^* wins G. Let us assume that $A \leq^b B^*$ and A wins G. We know that A^* wins G (Theorem 236) and $A^* \leq^i B$ (Theorem 218) and so $A^* = B$ (Definition 267) and so $A = B^*$ (taking the dual of both sides and using Theorem 75) and B^* is a minimal winning bipartition of G.

Theorem 80 tells us that the mapping is one-to-one. To show that it is onto, consider a minimal winning inverted biparition (B). B^* is a minimal winning bipartition. This is the preimage (under dual) of B. Because, of course, $(B^*)^* = B$ (Theorem 75).

□

Comments 270. The next few theorems explain how the minimal winning bipartitions and inverted bipartitions of G relate to those of $Dom_n(G)$ and $Cod_n(G)$.

Theorem 271. *A is a MWB of $Dom_n(G)$ iff $Dum_n(A)$ is a MWB of G*

Proof. A is a MWB of $Dom_n(G) \iff$

$\iff$ A wins $Dom_n(G)$ and ($B \leq^b A$ and B wins $Dom_n(G)$ implies $A = B$) (Definition 265)

Theorem 250 tells us that A wins $Dom_n(G)$ iff $Dum_n(A)$ wins G.

So now let us assume that $D \leq^b Dum_n(A)$ and D wins G. I need to show that $D = Dum_n(A)$

Theorem 171 and Theorem 190 tell us that D must be of the form $Dum_n(E)$ for E a bipartition.

Theorem 192 tells us that $E \leq^b A$.

$D = Dum_n(E)$ wins G. Theorem 250 tells us that E wins $Dom_n(G)$. Hence, by the minimality of A we can say that $E = A$ and so $Dum_n(E) = D = Dum_n(A)$.

Let us now say that $Dum_n(A)$ is a MWB of G.

Theorem 250 tells us that A wins $Dom_n(G)$.

$D \leq^b Dum_n(A)$ and D wins G, implies that $D = Dum_n(A)$ (Definition 265)

Assuming that $B \leq^b A$ and B wins $Dom_n(G)$, I need to show that $B = A$.

If $B \leq^b A$ then $Dum_n(B) \leq^b Dum_n(A)$ (Theorem 192)

B wins $Dom_n(G)$ implies that $Dum_n(B)$ wins G (Theorem 250)

And so, from the minimality of $Dum_n(A)$, $D = Dum_n(B) = Dum_n(A)$ and so (using Definition 83) $B = A$.

And we are done.

$\square$

Theorem 272. *A is a MWB of $Cod_n(G)$ $\implies$ $Vet_n(A)$ or $Dum_n(A)$ is a MWB of G.*

$Vet_n(A)$ is a MWB of G $\implies$ A is a MWB of $Cod_n(G)$

Proof. First, let us assume that A is a MWB of $Cod_n(G)$.

A wins $Cod_n(G)$ $\implies$ $Vet_n(A)$ wins G. (By Theorem 251.)

Does $Dum_n(A)$ win G? Let's assume that is does. I will show that $Dum_n(A)$ is a MWB of G.

Assume that $B \leq^b Dum_n(A)$ and B wins G. B must be of the from $Dum_n(C)$ where C is a bipartition (Theorem 171 and Theorem 190).

$Dum_n(C) \leq^b Dum_n(A)$ and Theorem 192 tell us that $C \leq^b A$.

$B = Dum_n(C)$ wins G and Theorem 250 tell us that C wins $Dom_n(G)$ and then Theorem 251 tells us that C wins $Cod_n(G)$.

A is a MWB of $Cod_n(G)$ so $C = A$. This tells us that $B = Dum_n(C) = Dum_n(A)$ and so $Dum_n(A)$ was a minimal winning bipartition.

Now, let us assume that $Dum_n(A)$ does not win G. We know that $Vet_n(A)$ does (Theorem 251). I will show that this is a minimal winning bipartition.

Let us assume that we have $B \leq^b Vet_n(A)$ and B wins G. B must be of the form $Dum_n(D)$ or $Vet_n(D)$ (Theorem 171).

It cannot be of the form $Dum_n(D)$ because if $B = Dum_n(D)$ wins G. Theorem 192 tells us that $D \leq^b A$ and $Dum_n(D) \leq Dum_n(A)$. Theorem 244 then tells us that $Dum_n(A)$ wins G which we know is not true.

So it must be of the form $Vet_n(D)$. Theorem 192 tell us that $D \leq^b A$.

$B = Vet_n(D)$ wins G and Theorem 251 tell us that D wins $Cod_n(G)$. A is a MWB of $Cod_n(G)$ so $D = A$, $Vet_n(D) = Vet_n(A)$ this shows that $Vet_n(A)$ is a MWB of G.

Now, let us assume that $Vet_n(A)$ is a MWB of G. I need to show that A is a MWB of $Cod_n(G)$.

First, Theorem 251 tells us that A wins $Cod_n(G)$.

Let as assume that $B \leq^b A$ and B wins $Cod_n(G)$. I need to show that $B = A$.

$B \leq^b A$ and Theorem 192 tell us that $Vet_n(B) \leq^b Vet_n(A)$.

B wins $Cod(G)$ and Theorem 251 tell us that $Vet(B)$ wins G.

$Vet(A)$ is a MWB of G and so $Vet(B) = Vet(A)$. Definition 88 tells us that $B = A$ and we are done.

$\square$

Theorem 273. *A is a MWIB of $Dom_n(G)$ iff $Dum_n(A)$ is a MWIB of G*

Proof. This is the dual of Theorem 271.

$\square$

Theorem 274. *A is a MWIB of $Cod_n(G) \implies Pas_n(A)$ or $Dum_n(A)$ is a MWIB of G.*

$Pas_n(A)$ is a MWIB of $G \implies A$ is a MWIB of $Cod_n(G)$

Proof. This is the dual of Theorem 272

$\square$

Comments 275. The two other implications that existed with winning do not exist with minimal winning.

If A is minimal winning for $Dom_n(G)$ then it is winning for $Cod_n(G)$ (Theorem 251) but we have no guarantee that it is minimal winning.

If $Dum_n(A)$ is minimal winning for G then $Vet_n(A)$ will be winning G (Theorem 250) but we know that it will not be minimal winning because $Dum_n(A) \leq^b Vet_n(A)$ (Theorem 197) and $Dum_n(A)$ wins. And, of course $Dum_n(A) \neq Vet_n(A)$ (Definition 88 and Definition 83).

The same implications apply to inverted bipartitions.

Theorem 276. *Every object of C_n is equal to the disjunction of its minimal winning bipartitions except for $\perp_n$ which can be thought of as the disjunction of no bipartitions*

Proof. C_0 has two objects and one minimal winning bipartition:$\top_0$

$\top_0$ is the disjunction of one bipartition: itself. $\perp_0$ is the disjunction of no bipartitions.

Let us say that the result holds in C_k.

Let G be an object in C_{k+1}.

The minimal winning bipartitions are of the form $Vet_{k+1}(A)$ or $Dum_{k+1}(B)$ where A and B are bipartitions (Theorem 171).

The disjunction of the MWBs of G is as follows:

$Vet_{k+1}(A_1) \vee Vet_{k+1}(A_2) \vee \cdots \vee Vet_{k+1}(A_r) \vee Dum_{k+1}(B_1) \vee Dum_{k+1}(B_2) \vee \ldots Dum_{k+1}(B_s)$. I need to show that this is equal to G. Definition 113 and Definitions 88 and 83 tell us that this is equal to

$Vet_{k+1}(A_1 \vee A_2 \vee \cdots \vee A_r) \vee Dum_{k+1}(B_1 \vee B_2 \vee \ldots B_s)$

The application of Dom_{k+1} to this gives us (Using Definition 92)

$B_1 \vee B_2 \vee \cdots \vee B_s$ (Clearly all of the $\perp_k$ from the $Vet_{k+1}(A_i)$ drop out of the disjunction by Theorem 118 and Theorem 104)

The application of Cod_{k+1} to the disjunction of MWBs of G gives us (Using Definition 94)

$A_1 \vee A_2 \vee \cdots \vee A_r \vee B_1 \vee B_2 \vee \cdots \vee B_s$

If I can show that $Dom_{k+1}(G) = B_1 \vee B_2 \vee \cdots \vee B_s$ and $Cod_{k+1}(G) = A_1 \vee A_2 \vee \cdots \vee A_r \vee B_1 \vee B_2 \vee \cdots \vee B_s$

Then I will have shown that $G = Vet_{k+1}(A_1) \vee Vet_{k+1}(A_2) \vee \cdots \vee Vet_{k+1}(A_r) \vee Dum_{k+1}(B_1) \vee Dum_{k+1}(B_2) \vee \ldots Dum_{k+1}(B_s)$ and I will have completed the induction.

$Dom_{k+1}(G) = B_1 \vee B_2 \vee \cdots \vee B_s$ is true by the induction hypothesis as each of the B_i is a MWB of $Dom_{k+1}(G)$ and they account for all the minimal winning bipartitions of $Dom_{k+1}(G)$. (By Theorem 271)

Can we say that same of $Cod_{k+1}(G) = A_1 \vee A_2 \vee \cdots \vee A_r \vee B_1 \vee B_2 \vee \cdots \vee B_s$?

Theorem 272 tells us that the list of A_i and B_i includes all the minimal winning bipartitions of $Cod_{k+1}(G)$.

All of the A_i are MWBs of $Cod_{k+1}(G)$ (Theorem 272). All of the B_i are winning bipartitions of $Dom_{k+1}(G)$ (Theorem 250) and so they are winning bipartitions of $Cod_{k+1}(G)$ (Theorem 251) but it is possible that they are not minimal winning bipartitions of $Cod_{k+1}(G)$. However I will now show that these drop out of the disjunction. Each has a MWB that is smaller than it with respect to $\leq^b$ (Theorem 266). If D is a MWB with $D \leq^b B$ (B a winning bipartition) then $B \leq D$ (Theorem 240. And so $B \vee D = D$. This tells us that the B_i that are not minimal winning just drop out of the disjunction so we are left with only minimal winning bipartitions. And so $A_1 \vee A_2 \vee \cdots \vee A_r \vee B_1 \vee B_2 \vee \cdots \vee B_s$ is equal to the disjunction of the MWBs of $Cod_{k+1}(G)$ and we are done by the induction hypothesis.

$\square$

Theorem 277. *Every object of C_n is equal to the conjunction of its minimal winning inverted bipartitions except for $\top_n$ which can be thought of as the conjunction of no inverted bipartitions*

Proof. This is the dual of Theorem 276.

$\square$

Comments 278. C_n is a category but not quite a Boolean Algebra. The set of bipartitions and the set of inverted bipartitions in C_n (From now on, I will refer to them as B_n and I_n) provide the objects for two Boolean Algebras and also provide the objects for an (order) category.

4.6 The Category of Bipartitions

Definition 279. *The category of bipartitions, B_n* has as objects the bipartitions in C_n. If E and F are objects then there is an arrow from E to F iff $E \leq^b F$.

Theorem 280. *The category of bipartitions is a category.*

Proof. $\leq^b$ is a partial order (Theorem 196). The fact that this defines a category is a standard result in category theory.

□

Definition 281. *The category of inverted bipartitions, I_n* has objects that are inverted bipartitions in C_n. If E and F are objects then there is an arrow from E to F iff $E \leq^i F$.

Theorem 282. *The category of inverted bipartitions is a category.*

Proof. $\leq^i$ is a partial order (Theorem 222). The fact that this defines a category is a standard result in category theory.

□

Comments 283. We already have many limits for these categories

Theorem 284. $\top_n$ *is initial in the category of bipartitions*

Proof. $\top_n$ is minimal in the category of bipartitions (Theorem 201). So there is an arrow from $\top_n$ to any object in the bipartition category (Definition 279). There cannot be more than one arrow between any two objects in an order category.

$\square$

Theorem 285. *$\top_n^c$ is terminal in the category of bipartitions*

Proof. $\top_n^c$ is maximal in the category of bipartitions. $\top_n$ is minimal (Theorem 201) and taking the complement flips the inequality (Theorem 198). So there is an arrow from any object to $\top_n^c$ in the bipartition category (Definition 279). There cannot be more than one arrow between any two objects in an order category.

$\square$

Theorem 286. *If A and B are objects of the bipartition category then $A \wedge^b B$ is the product of A and B.*

Proof. $A \wedge^b B$ is the greatest lower bound of A and B (Theorem 200) and so it is less than A and B and hence there are arrows from it to A and B (Definition 279). Of course, these are the projections.

Let us say that there is a U with arrows from U to A and B. This tells us that $U \leq^b A$ and $U \leq^b B$ (Definition 279). Since $A \wedge^b B$ is the greatest lower bound (Theorem 200), $U \leq^b A \wedge^b B$ and there is an arrow from U to $A \wedge^b B$ (Definition 279). This commutes with the projections because there is not more than one arrow between any two objects in an order-category.

$\square$

Theorem 287. *If A and B are objects of the bipartition category then $A \vee^b B$ is the coproduct of A and B.*

Proof. $A \vee^b B$ is the least upper bound of A and B (Theorem 199) and so it is greater than A and B and hence there are arrows from A and B to it (Definition 279). Of course, these are the coprojections.

Let us say that there is a U with arrows from A and B to U. This tells us that $A \leq^b U$ and $B \leq^b U$ (Definition 279). Since $A \vee^b B$ is the least upper bound (Theorem 199), $A \vee^b B \leq^b U$ and there is an arrow from $A \vee^b B$ to U (Definition 279). This commutes with the coprojections because there is not more than one arrow between any two objects in an order-category.

□

Theorem 288. *$\perp_n$ is initial in the category of inverted bipartitions*

Proof. This is the dual of Theorem 284.

□

Theorem 289. *$\perp_n^c$ is terminal in the category of inverted bipartitions*

Proof. This is the dual of Theorem 285.

□

Theorem 290. *If A and B are objects of the inverted bipartition category then $A \wedge^i B$ is the product of A and B.*

Proof. This is the dual of Theorem 286.

□

Theorem 291. *If A and B are objects of the inverted bipartition category then $A \vee^i B$ is the coproduct of A and B.*

Proof. This is the dual of Theorem 287.

□

4.7 Defining the Voters

Comments 292. We have got this far without defining the voters in these bipartitions. Since the bipartitions are a finite Boolean Algebra, with 2^n elements, they must be isomorphic to the Boolean Algebra of finite sets and the n atoms correspond to the n singletons.

Definition 293. I define the atoms of the Boolean Algebra B_n as *voters.* Or more precisely, the atoms in the Boolean Algebra correspond to singleton sets each of which contains one of the voters.

Definition 294. I define the atoms of the Boolean Algebra I_n as *inverted voters.* Or more precisely, the atoms in the Boolean Algebra correspond to singleton sets each of which contains one of the voters.

Theorem 295. *There are n atoms of B_n. These are: $Dict_n$, $Dum_n(Dict_{n-1})$, $Dum_n(Dum_{n-1}((Dict_{n-2}))$, $\ldots$, $Dum_n(Dum_{n-1}(\ldots Dum_1(Dict_1))$.*

I will refer to them as $v_{n,n}, v_{n,n-1}, \ldots, v_{n,1}$ respectively.

Proof. The proof is by induction on n. B_1 has one non-zero object: $Dict_1$. This is the only atom.

Let us say that we have the theorem for $n = k$.

What are the atoms in B_{k+1}?

$Dum_{k+1}(v_{k,i})$, (which I am calling $v_{k+1,i}$ for all $i = 1, \ldots, k$), is an atom of B_{k+1}. Let us show this. First, it is not equal to $\top_{k+1}$ (Using Definition 83 and Definition 85), which is the zero of the Boolean Algebra.

Next, let us consider $W \leq^b Dum_{k+1}(v_{k,i})$. Theorem 171 tells us that any bipartition, W, is of the form $Vet_{k+1}(U)$ or $Dum_{k+1}(U)$ with U a bipartition of C_k.

Also $W \leq^b Dum_{k+1}(v_{k,i})$ tells us that W must be of the form $Dum_{k+1}(U)$, by Theorem 190.

So $W \leq^b Dum_{k+1}(v_{k,i})$

$\implies Dum_{k+1}(U) \leq^b Dum_{k+1}(v_{k,i})$

$\implies U \leq^b v_{k,i}$ (Theorem 192)

Since $v_{k,i}$ is an atom.

$U = \top_k$ or $U = v_{k,i}$

This means that $W = Dum_{k+1}(\top_k) = \top_{k+1}$ (Using Definition 85)

Or $W = Dum_{k+1}(v_{k,i})$.

So we have shown that all of the $Dum_{k+1}(v_{k,i}) = v_{k+1,i}$ are atoms.

I will now show that $Dict_{k+1}$ is an atom. It is not equal to $\top_{k+1}$ (Definition 110).

Let us say that $W \leq^b Dict_{k+1}$.

We know, from Theorem 171, that W is equal to $Vet_{k+1}(U)$ or $Dum_{k+1}(U)$ with U a bipartition.

$Dict_{k+1} = Vet_k(\top_k)$ (Definition 110)

Theorem 192 tells us that $Vet_k(U) \leq^b Vet_k(\top_k) \implies U \leq^b \top_k$. Theorem 240 then gives us $U \geq \top_k$. Theorem 105 then tells us that $U = \top_k$ and $W = Dict_{k+1}$.

Theorem 192 tells us that $Dum_k(U) \leq^b Vet_k(\top_k) \implies U \leq^b \top_k$. Theorem 240 then gives us $U \geq \top_k$. Theorem 105 then tells us that $U = \top_k$ and $W = \top_{k+1}$ (Using Definition 85).

This shows that $Dict_{k+1}$ is an atom. I call it $v_{k+1,k+1}$.

Now let's us assume that U is an atom. I need to show that it is equal to one of the $\{v_{k+1,i} : 1 \leq i \leq k+1\}$.

U is a bipartition and so it must be of the form $Vet_{k+1}(W)$ or $Dum_{k+1}(W)$ with W a bipartition. (Theorem 171)

If $U = Vet_{k+1}(W)$ then we know $Dum_{k+1}(W) \leq^b Vet_{k+1}(W)$ (Theorem 197). If U is to be an atom then we need $Dum_{k+1}(W)$ equal to $\top_{k+1}$ and $W = \top_k$ (Definition 85). This would make U equal to $Dict_{k+1}$ (Definition 110).

If $U = Dum_{k+1}(W)$ then W must be an atom in B_k because if there is an $X \leq^b W$ then (Theorem 192) $Dum_{k+1}(X) \leq^b Dum_{k+1}(W)$. For $Dum_{k+1}(W)$ to be an atom, we need $Dum_{k+1}(X)$ to equal $\top_{k+1}$ (or $Dum_{k+1}(W)$) and that requires X to be $\top_k$ or W. So W is an atom of B_k and hence it is of the form $Dict_k$, $Dum_k(Dict_{k-1})$, $Dum_k(Dum_{k-1}((Dict_{k-2}))$, $\ldots, Dum_k(Dum_{k-1}(\ldots Dum_1(Dict_1))$. and so $Dum_{k+1}(W)$ is of the form $Dum_{k+1}(Dict_k)$, $Dum_{k+1}(Dum_k((Dict_{k-1}))$, $\ldots$, $Dum_{k+1}(Dum_k(\ldots Dum_1(Dict_1))$

$\square$

Theorem 296. *There are n atoms of I_n. These are: $Dict_n$, $Dum_n(Dict_{n-1})$, $Dum_n(Dum_{n-1}((Dict_{n-2}))$, $\ldots$, $Dum_n(Dum_{n-1}(\ldots Dum_1(Dict_1))$. Looking back at Theorem 295, we see that this means that that B_n and I_n both have the same atoms.*

Proof. This is the dual of Theorem 295.

$\square$

Theorem 297. *The atoms of B_i are equal to their own dual.*

Proof. This is immediate from Theorem 138 and Theorem 101.

$\square$

Theorem 298. *In C_n, the $v_{n,i}$ (As i goes from 1 to n) are the only objects that are bipartitions and inverted bipartitions*

Proof. By Theorem 295 and Theorem 296, the $v_{n,i}$ are bipartitions and inverted bipartitions, in fact they are the atoms of these Boolean Algbras.

Let y be a bipartition and an inverted bipartition. The $v_{n,i}$ are atoms and so there must be a $v_{n,j}$ such that $v_{n,j} \leq^b y$. This is equivalent to $v_{n,j} \geq y$ (By Theorem 240) or $v_{n,j} \geq^i y$ (By Theorem 243). $v_{n,j}$ is an atom of the Boolean Algebra of inverted bipartitions. This makes y an atom of the Boolean Algebra of inverted bipartitions or $\perp$. It can't be the later as this is not a bipartition (Definition 145)

□

Comments 299. To fully develop the isomorphism between the Boolean Algebra of bipartitions and the Boolean Algebra of finite sets, we need to know what it is for a voter to vote positively in a biparition. This corresponds to the voter being a member of a set.

Definition 300. A voter $v_{k,i}$ votes 'yes' in a bipartition B iff there is an arrow from B to $v_{k,i}$

Definition 301. A voter $v_{k,i}$ votes 'yes' in an inverted bipartition B iff there is an arrow from $v_{k,i}$ to B

Theorem 302. *$v_{k,i}$ votes 'yes' in a bipartition B iff it votes 'yes' in the inverted bipartition B^*.*

Proof. $v_{k,i}$ votes 'yes' in B

$\iff$ There is an arrow from B to $v_{k,i}$ (Definition 300)

$\iff$ There is an arrow form $v_{k,i}$ to B^* (Taking the dual and using Definition 72 and Definition 66)

$\iff$ $v_{k,i}$ votes 'yes' in B^*. (Definition 301)

$\square$

Theorem 303. *Every bipartition, B, in C_n is of the form $Vet_n(C)$ or $Dum_n(C)$ where C is a bipartition (Theorem 171).*

$Dict_n = v_{n,n}$ votes 'yes' in B iff B is of the form $Vet_n(C)$.

Proof. $B = Vet_n(C) = [\bot_{n-1}, C]$ (Definition 88) and $Dict_n = [\bot_{n-1}, \top_{n-1}]$.

There is an identity arrow from $\bot_{n-1}$ to itself. There is also an arrow from C to $\top_{n-1}$ because $\top_{n-1}$ is terminal (Theorem 103). These two together give us an arrow from $Vet_n(C)$ to $Dict_n$ (Theorem 70). So $Dict_n$ votes 'yes' in $Vet_n(C) = B$ (Definition 300)

$Dict_n$ cannot vote 'yes' in $Dum_n(C)$ because that would require an arrow from $Dum_n(C)$ to $Dict_n$ (Definition 300). In turn, this would require arrows from C to $\bot_{k-1}$ and C to $\top_{k-1}$ (Theorem 70). There would only be an arrow from C to $\bot_{k-1}$ if C was $\bot_{k-1}$ (Theorem 104). In this case $Dum_n(\bot_{k-1}) = \bot_k$ (Definition 85) is not a bipartition (Definition 145)

$\square$

Theorem 304. *Every inverted bipartition, I, in C_n is of the form $Pas_n(C)$ or $Dum_n(C)$ where C is an inverted bipartition (Theorem 173).*

$Dict_n = v_{n,n}$ votes 'yes' in I iff I is of the form $Pas_n(C)$.

Proof. This is the dual of Theorem 303.

$\square$

Theorem 305. *Let B be an object of B_{k-1}.*

If $r \leq k-1$ then $v_{k,r}$ votes 'yes' in $Dum_k(B)$ in C_k iff $v_{k-1,r}$ votes 'yes' in B in C_{k-1}

If $r \leq k-1$ then $v_{k,r}$ votes 'yes' in $Vet_k(B)$ in C_k iff $v_{k-1,r}$ votes 'yes' in B in C_{k-1}

Proof. $v_{k,r}$ votes 'yes' in $Dum_k(B)$ in C_k

$\iff$ There is an arrow from $Dum_k(B)$ to $v_{k,r}$ in C_k (Definition 300)

$\iff$ There is an arrow from B to $v_{k-1,r}$ in C_{k-1} (Definition 83 and Theorem 70)

$\iff$ $v_{k-1,r}$ votes 'yes' in B in C_{k-1}. (Definition 300)

$v_{k,r}$ votes 'yes' in $Vet_k(B)$ in C_k

$\iff$ There is an arrow from $Vet_k(B)$ to $v_{k,r}$ in C_k (Definition 300)

$\iff$ There is an arrow from B to $v_{k-1,r}$ in C_{k-1} (Definition 88 and Theorem 70)

$\iff$ $v_{k-1,r}$ votes 'yes' in B in C_{k-1} (Definition 300).

□

Theorem 306. *Let I be an object of I_n.*

If $r \le k-1$ then $v_{k,r}$ votes 'yes' in $Dum_k(I)$ in C_k iff $v_{k-1,r}$ votes 'yes' in I in C_{k-1}

If $r \le k-1$ then $v_{k,r}$ votes 'yes' in $Pas_k(I)$ in C_k iff $v_{k-1,r}$ votes 'yes' in I in C_{k-1}

Proof. This is the dual of Theorem 305.

□

Theorem 307. *Let B be a bipartition in C_k. For all $r \le k$,$v_{k,r}$ votes 'yes' in B iff it votes 'no' in B^c.*

Proof. $v_{k,k} = Dict_k$ votes 'yes' in B

$\iff$ B is of the form $Vet_k(D)$ where D is a bipartition (Theorem 303)

$\iff$ B^c is of the form $Dum_k(D^c)$ (Definition 176)

$\iff$ $v_{k,k} = Dict_k$ votes 'no' in B^c. (Theorem 303)

Let us say that $v_{k,k-1}$ votes 'yes' in B in C_k. We have two cases: $B = \mathit{Dum}_k(D)$ and $B = \mathit{Vet}_k(D)$.

If $B = \mathit{Dum}_k(D)$ then.

$v_{k,k-1}$ votes 'yes' in $B = \mathit{Dum}_k(D)$

$\iff$ $v_{k-1,k-1}$ votes 'yes' in D (Theorem 305)

$\iff$ $v_{k-1,k-1}$ votes 'no' in D^c (As shown earlier in this proof)

$\iff$ $v_{k,k-1}$ votes 'no' in $\mathit{Vet}_k(D^c)$ (Theorem 305)

$\iff$ $v_{k,k-1}$ votes 'no' in $\mathit{Dum}_k(D)^c$ (Definition 176)

If $B = \mathit{Vet}_k(D)$ then.

$v_{k,k-1}$ votes 'yes' in $B = \mathit{Vet}_k(D)$

$\iff$ $v_{k-1,k-1}$ votes 'yes' in D (Theorem 305)

$\iff$ $v_{k-1,k-1}$ votes 'no' in D^c (As shown earlier in this proof)

$\iff$ $v_{k,k-1}$ votes 'no' in $\mathit{Dum}_k(D^c)$ (Theorem 305)

$\iff$ $v_{k,k-1}$ votes 'no' in $\mathit{Vet}_k(D)^c$ (Definition 176)

Applying this sort of reasoning r times, we can show that $v_{k,k-r}$ votes 'yes' in $B \iff v_{k,k-r}$ votes 'no' in B^c for all r from 1 to k.

□

Theorem 308. *Let I be an inverted bipartition in C_k. For all $r \leq k$,v_r votes 'yes' in I iff it votes 'no' in I^c.*

Proof. This is the dual of Theorem 307.

□

Theorem 309. *In C_n, $v_{n,i}$ votes 'yes' in the bipartition $C \vee^b D$ if and only if it votes 'yes' in the bipartition C or votes 'yes' in the bipartition D.*

Proof. $v_{n,i}$ votes 'yes' in C

$\implies$ There is an arrow from C to $v_{n,i}$ (Definition 300)

$\implies$ There is an arrow from $C \wedge D$ to $v_{n,i}$ (Combining this with the projection map from $C \wedge D$ to C.)

$\implies$ There is an arrow from $C \vee^b D$ to $v_{n,i}$ (Theorem 238)

$\implies$ $v_{n,i}$ votes 'yes' in $C \vee^b D$ (Definition 300)

The proof that $v_{n,i}$ votes 'yes' in $D \implies v_{n,i}$ votes 'yes' in $C \vee^b D$ is similar.

The proof that $v_{n,i}$ votes 'yes' in $C \vee^b D$ implies $v_{n,i}$ votes 'yes' in C is by induction. This must be true in C_0 because there is only one bipartition: $\top_0$ and so C will always be equal to $C \vee^b D$ in C_0

Let us say that the theorem holds in C_{k-1}

Now let us say that $v_{k,i}$ votes 'yes' in $C \vee^b D$ in C_k.

If $v_{k,i} = Dict_k$ then

$Dict_k$ votes 'yes' in $C \vee^b D$

$\implies$ There is an arrow from $C \vee^b D$ to $Dict_k$ (Definition 300)

$\implies$ $Dom_k(C \vee^b D) = \perp_{n-1}$ (Definition 110, Definition 66 and Theorem 104)

$\implies$ $Dom_k(C \wedge D) = \perp_{n-1}$ (Theorem 238)

$\implies$ $Dom_k(C) \wedge Dom_k(D) = \perp_{n-1}$ (Definition 111)

$\implies$ $Dom_k(C)$ or $Dom_k(D)$ is equal to $\perp_{n-1}$ (Theorem 104 and Theorem 112)

Without loss of generality $Dom_k(C) = \perp_{n-1}$

$\implies$ There is an arrow from C to $Dict_k$ (Definition 110 and Definition 92)

$\implies$ $Dict_k = v_{k,i}$ votes 'yes' in C (Definition 300)

If $v_{k,i} = Dum_k(v_{k-1,i})$

$v_{k,i}$ votes 'yes' in $C \vee^b D$

$\implies$ There is an arrow from $C \vee^b D$ to $v_{k,i}$ (Definition 300)

$\implies$ There is an arrow from $C \wedge D$ to $v_{k,i}$ (Theorem 238)

$\implies$ There is an arrow from $Dom_k(C \wedge D)$ to $v_{k-1,i}$ and an arrow from $Cod_k(C \wedge D)$ to $v_{k-1,i}$ (Theorem 70)

$\implies$ There is an arrow from $Dom_k(C) \wedge Dom_k(D)$ to $v_{k-1,i}$ and an arrow from $Cod_k(C) \wedge Cod_k(D)$ to $v_{k-1,i}$ (Definition 111)

$\implies$ There is an arrow from $Dom_k(C) \vee^b Dom_k(D)$ to $v_{k-1,i}$ and an arrow from $Cod_k(C) \vee^b Cod_k(D)$ to $v_{k-1,i}$ (Theorem 238)

$\implies$ $v_{k-1,i}$ votes 'yes' in $Dom_k(C) \vee^b Dom_k(D)$ and votes 'yes' in $Cod_k(C) \vee^b Cod_k(D)$ (Definition 300)

$\implies$ $v_{k-1,i}$ votes 'yes' in $Cod_k(C)$ or $Cod_k(D)$ (By the Induction Hypothesis)

Without loss of generality $v_{k-1,i}$ votes 'yes' in $Cod_k(C)$

$\implies$ There is an arrow from $Cod_k(C)$ to $v_{k-1,i}$ (Definition 300)

$\implies$ There is an arrow from $Dom_k(C)$ to $v_{k-1,i}$ (combining this with the arrow from $Dom(C)$ to $Cod(C)$ (Theorem 68))

$\implies$ There is an arrow from C to $v_{k,i}$ (Theorem 70).

$\implies$ $v_{k,i}$ votes 'yes' in C. (Definition 300).

This completes the induction.

$\square$

Theorem 310. *In C_n, $v_{n,i}$ votes 'yes' in $C \wedge^b D$ if and only if it votes 'yes' in C and $v_{n,i}$ votes 'yes' in D.*

Proof. $v_{n,i}$ votes 'yes' in $C \wedge^b D$

$\iff v_{n,i}$ votes 'no' in $(C \wedge^b D)^c$ (Theorem 307)

$\iff v_{n,i}$ votes 'no' in $C^c \vee^b D^c$ (Theorem 187)

$\iff$ It is not the case that ($v_{n,i}$ votes 'yes' in C^c or $v_{n,i}$ votes 'yes' in D^c) (Theorem 309)

$\iff v_{n,i}$ votes 'no' in C^c and $v_{n,i}$ votes 'no' in D^c

$\iff v_{n,i}$ votes 'yes' in C and $v_{n,i}$ votes 'yes' in D (Theorem 307).

□

Theorem 311. *In C_n, $v_{n,i}$ votes 'yes' in the inverted bipartition $C \vee^i D$ if and only if it votes 'yes' in the inverted bipartition C or votes 'yes' in the inverted bipartition D.*

Proof. This is the dual of Theorem 309.

□

Theorem 312. *In C_n, $v_{n,i}$ votes 'yes' in the inverted bipartition $C \wedge^i D$ if and only if it votes 'yes' in the inverted bipartition C and votes 'yes' in the inverted bipartition D.*

Proof. This is the dual of Theorem 310.

□

Theorem 313. *Every bipartition in B_k can be written as a conjunction of the $v_{k,i}$, except for $\top_k$ which we can think of as the conjunction of no $v_{k,i}$s.*

Proof. The result is true for $k = 0$: $\top_0$ is the only bipartition in B_0.

It is also true for $k = 1$. There are only two bipartitions in B_1: $Dict_1 = v_{1,1}$ and $\top_1$.

Let us assume that the theorem is true for $k = n$.

Consider a bipartition in B_{k+1}. Theorem 171 tells us that it must be of the form $Dum_{k+1}(B)$ or $Vet_{k+1}(B)$ with B a bipartition.

By the induction hypothesis (and Theorem 295), B is a conjunction of the $Dict_k$, $Dum_k(Dict_{k-1})$, $Dum_k(Dum_{k-1}((Dict_{k-2}))$, $\ldots, Dum_k(Dum_{k-1}(\ldots Dum_1(Dict_1)))$. Applying Dum_{k+1} to this and using Theorem 400 (Dum_{k+1} preserves $\vee$) we can see that $Dum_{k+1}(B)$ is a conjunction of the

$Dum_{k+1}(Dict_k)$, $Dum_{k+1}(Dum_k(Dict_{k-1}))$, $Dum_{k+1}(Dum_k(Dum_{k-1}((Dict_{k-2})))$, $\ldots Dum_{k+1}(Dum_k(Dum_{k-1}(\ldots Dum_1(Dict_1))))$

By direct calculation $Vet_{k+1}(B) = Dict_{k+1} \wedge Dum_{k+1}(B)$ (Definition 88, Definition 83, Definition 110 and Theorem 104).

Above, we saw that $Dum_{k+1}(B)$ is a conjunction of the $Dum_{k+1}(Dict_k)$, $Dum_{k+1}(Dum_k(Dict_{k-1}))$, $Dum_{k+1}(Dum_k(Dum_{k-1}((Dict_{k-2})))$, $\ldots$ $Dum_{k+1}(Dum_k(Dum_{k-1}(\ldots Dum_1(Dict_1))))$. This means that $Vet_{k+1}(B)$ is a conjunction of the $Dict_{k+1}$, $Dum_{k+1}(Dict_k)$, $Dum_{k+1}(Dum_k(Dict_{k-1}))$, $Dum_{k+1}(Dum_k(Dum_{k-1}((Dict_{k-2})))$, $\ldots, Dum_{k+1}(Dum_k(Dum_{k-1}(\ldots Dum_1(Dict_1))))$.

By Theorem 295 $Vet_{k+1}(B)$ is a conjunction of the $v_{n,1} \ldots v_{n,n}$

This completes the induction.

□

Theorem 314. *Every inverted bipartition in I_k can be written as a disjunction of the $v_{k,i}$, except for $\perp_k$ which we can think of as the disjunction of no $v_{k,i}$s.*

Proof. This is the dual of Theorem 313.

□

Theorem 315. *If B is a bipartition in C_n then B can be of the form $Dum_n(D)$ or $Vet_n(D)$ with D a bipartition*

$Dict_n = v_{n,n}$ appears in any expression of B as a conjunction of the voters if and only if B is of the form $Vet_n(D)$

Proof. B is of the form $Vet_n(D) \implies$

$Dom_n(B) = \perp_{n-1} \implies$

$Dict_n$ is in the conjunction. If not then B would be a conjunction of the $Dum_n(Dict_{n-1})$, $Dum_n(Dum_{n-1}(Dict_{n-2}))$, $Dum_n(Dum_{n-1}(Dum_{n-2}((Dict_{n-3})))$, $\ldots$, $Dum_n(Dum_{n-1}(Dum_{n-2}(\ldots Dum_1(Dict_1))))$.

In this case (Definition 111) B is Dum_n of a conjunction of the $Dict_{n-1}$, $Dum_{n-1}(Dict_{n-2})$, $Dum_{n-1}(Dum_{n-2}((Dict_{n-3}))$, $\ldots$, $Dum_{n-1}(Dum_{n-2}(\ldots Dum_1(Dict_1)))$. Dom_n of this is a conjunction of the $Dict_{n-1}$, $Dum_{n-1}(Dict_{n-2})$, $Dum_{n-1}(Dum_{n-2}((Dict_{n-3}))$, $\ldots$, $Dum_{n-1}(Dum_{n-2}(\ldots Dum_1(Dict_1)))$ (By Definition 83 and Definition 92).

This cannot be $\perp_{n-1}$ because none of the objects in the conjunction are $\perp_{n-1}$ (Using Theorem 130)

Now, let us say that $Dict_n$ is in the expression of B as a conjunction of voters. Dom_n of the conjunction is just the conjunction of Dom_n acting on the various terms (Definition 111). $Dom_n(Dict_n) = \perp_{n-1}$ (Definition 88). Theorem 130 tells us that $Dom_n(B)$ of the conjunction is equal to $\perp_{n-1}$. And so B must be $Vet_n(D)$. If it was $Dum_n(D)$, then we would need D to be $\perp_{n-1}$ which is not possible because $\perp_{n-1}$ is not a bipartition (Definition 145)

□

Theorem 316. *If I is an inverted bipartition in C_n then I can be of the* *form $Dum_n(J)$ or $Pas_n(J)$ with J an inverted bipartition*

$Dict_n$ appears in any expression of I as a disjunction of the voters if and only if I is of the form $Pas_n(J)$

Proof. This is the dual of Theorem 315.

□

Theorem 317. *The expression of a bipartition, in C_n, as a conjunction of the voters is unique.*

Proof. The proof is by induction on n. In C_0, there is only one bipartition: $\top_0$ that is the conjunction of no voters.

Let us say that the theorem holds for $n = k$.

Let B be a bipartition in C_{k+1}. B could be of the form $Vet_{k+1}(D)$ or $Dum_{k+1}(D)$ with D a bipartition in C_k (Theorem 171).

First, let us say that B is of the form $Dum_{k+1}(D)$. $Dict_{k+1} = v_{k+1,k+1}$ cannot be in the conjunction. $B = Dum_{k+1}(D)$ is a conjunction of the $Dum_{k+1}(Dict_k)$, $Dum_{k+1}(Dum_k((Dict_{k-1}))$, $\ldots$, $Dum_{k+1}(Dum_k(\ldots Dum_1(Dict_1)))$. Let us say that it could be expressed as two different conjunctions of these (i.e. not two conjunctions in which they appeared in a different order or one of them appeared multiple times but two conjunctions for which one of these voters was in one conjunction and not in the other.). Definition 111 tells us that D can be written as two different conjunctions of the $Dict_k$, $Dum_k((Dict_{k-1})$, $\ldots$, $Dum_k(\ldots Dum_1(Dict_1))$. This would contradict the induction hypothesis.

If B is of the form $Vet_{k+1}(D)$ then $Dict_{k+1}$ is in the conjunction (Theorem 315).

Direct calculation shows us that $Vet_{k+1}(D) = Dict_{k+1} \wedge Dum_{k+1}(D)$. We know that the conjunction expression of $Dum_{k+1}(D)$ is unique so the expression of $Vet_{k+1}(D)$ as a conjunction must also be unique.

$\square$

Theorem 318. *The expression of an inverted bipartition, in C_n, as a disjunction of the voters is unique.*

Proof. This is the dual of Theorem 317.

$\square$

Theorem 319. *Let D be a bipartition in C_n. $Dum_n(D)$ and $Vet_n(D)$ are also biparitions (By Theorem 171)*

$\forall i : 1 \leq i \leq n$, $v_{n+1,i}$ is in the representation of $Dum_{n+1}(D)$ as a conjunction iff $v_{n,i}$ is in the representation of D as a conjunction

$\forall i : 1 \leq i \leq n$, $v_{n+1,i}$ is in the representation of $Vet_{n+1}(D)$ as a conjunction iff $v_{n,i}$ is in the representation of D as a conjunction

Proof. Let us say that we have a representation of D as a conjunction of the $v_{n,i}$ with $1 \leq i \leq n$.

$v_{n+1,i} = Dum_{n+1}(v_{n,i})$ (Theorem 295) and so (By Definition 111 and Definition 83) if $v_{n,i}$ is in the conjunction representation of D then $v_{n+1,i}$ is in the conjunction representation of $Dum_{n+1}(D)$.

To prove the implication in the other direction, we start from $Dom_{n+1}(A \wedge B) = Dom_{n+1}(A) \wedge Dom_{n+1}(B)$ (Definition 111).

$Dom_{n+1}(Dum_n(D)) = D$ (Definition 83 and Definition 92) and for all $i \leq n$, $Dom_{n+1}(v_{n+1,i}) = v_{n,i}$. Dom_{n+1} respects wedge (Definition 111 and Definition 92)

And so if $v_{n+1,i}$ is in the conjunction representation of $Dum_{n+1}(D)$ then $Dom(v_{n+1,i}) = v_{n,i}$ is in the conjunction representation of $Dom_{n+1}(Dum_n(D)) = D$

Let us say that we have a representation of D as a conjunction of the $v_{k,i}$.

Direct calculation show us that $Vet_n(D) = Dict_n \wedge Dum_n(D)$. (Definitions 83, 88 and 110).

By the above argument, if $v_{n,i}$ is in the (unique! Theorem 317) conjunction of D then $v_{n+1,1}$ is in the (unique) conjunction representation of $Dum_n(D)$ and hence, by the line above, it is in the (unique) conjunction representation of $Vet_n(D)$.

To prove the implication in the other direction, $Dom_n(A \wedge B) = Dom_n(A) \wedge Dom_n(B)$ (Definition 111).

$Cod_{n+1}(Vet_n(D)) = D$ (Definition 88 and Definition 94) and for all $i \leq k$, $Cod_{n+1}(v_{n+1,i}) = v_{n,i}$. Cod_n respects wedge (Definition 94 and Definition 111)

And so if $v_{n+1,i}$ is in the (unique! Theorem 317) conjunction representation of $Vet_n(D)$ then $Cod_{n+1}(v_{n+1,i}) = v_{n,i}$ is in the (unique) conjunction representation of $Cod_{n+1}(Vet_n(D)) = D$

□

Theorem 320. *Let D be an inverted bipartition in C_n. $Dum_n(D)$ and $Pas_n(D)$ are also inverted biparitions (By Theorem 173)*

$\forall i : 1 \leq i \leq n$, $v_{n+1,i}$ is in the representation of $Dum_n(D)$ as a disjunction iff $v_{n,i}$ is in the representation of D as a disjunction

$\forall i : 1 \leq i \leq n$, $v_{n+1,i}$ is in the representation of $Pas_n(D)$ as a disjunction iff $v_{n,i}$ is in the representation of D as a disjunction

Proof. This is the dual of Theorem 319.

□

Theorem 321. *Let B be a bipartition. $v_{n,i}$ votes 'yes' in $B \iff v_{n,i}$ appears in the expression of B as a conjunction of the $v_{n,r}$.*

Proof. $v_{n,i}$ appears in the conjunction

$\implies B \leq v_{n,i}$ (Theorem 112)

$\implies$ There is an arrow from B to $v_{n,i}$ (Definition 66)

$\implies v_{n,i}$ votes 'yes' in B (Definition 300)

The proof in the other direction is by induction.

In C_1 there are two bipartitions: $v_{1,1}$ and $\top_1$ and one atom: $v_{1,1}$.

$v_{1,1}$ votes 'yes' in $v_{1,1}$. The arrow from $v_{1,1}$ to $v_{1,1}$ is the identity.

$v_{1,1}$ is not in the expansion of $\top_1$ as a conjunction and there is no arrow from $\top_1$ to $v_{1,1}$.

Let us assume that the theorem holds in C_k

Now $v_{k+1,i}$ could be $Dict_{k+1}$ or $Dum_{k+1}(v_{k,i})$ (By Theorem 171)

Let us start with the case that $v_{k+1,i} = Dict_{k+1}$

$v_{k+1,i}$ votes 'yes' in B

$\implies$ There is an arrow from B to $Dict_{k+1}$ (Definition 300)

$\implies Dom_{k+1}(B) = \perp_k$ (Definition 110, Theorem 104 and Definition 66)

$\implies B$ is of the form $Vet_{k+1}(D)$ (Definition 88)

$\implies Dict_{k+1}$ is in the representation of $B = Vet_{k+1}(D)$ as a conjunction (Theorem 315)

Now let us move on to the case, $v_{k+1,i} = Dum_{k+1}(v_{k,i})$

$v_{k+1,i}$ votes 'yes' in B

$\implies$ There is an arrow from B to $Dum_{k+1}(v_{k,i})$ (Definition 300)

$\Longrightarrow$ There is an arrow from $Cod_k(B)$ to $v_{k,i}$ (Definition 94 and Definition 83)

$\Longrightarrow$ $v_{k,i}$ votes 'yes' in $Cod_k(B)$ (Definition 300)

$\Longrightarrow$ $v_{k,i}$ is in the conjunction expansion of $Cod_k(B)$. (Induction hypothesis)

Theorem 319 tells us (By Theorem 171 B must be $Dum_{k+1}(Cod_k(B))$ or $Vet_{k+1}(Cod_k(B)))$ that $v_{k+1,i}$ is in the conjunction expansion of B.

□

Comments 322. This now gives us an isomorphism between the Boolean Algebra B_n an the Boolean Algebra of subsets of the finite set $\{1, \ldots, n\}$.

Definition 323. $\Omega : B_n \to 2^{\{1,\ldots,n\}}$.

Let D be an object of B_n then I define $\Omega(D)$ to be the set of all r such that v_r is in the expression of D as a conjunction

Theorem 313 and Theorem 317 tells us that this is possible.

Theorem 324. *Ω is one-to-one and onto.*

Proof. Every conjunction of the voters is a bipartition because the voters are bipartitions. They are atoms in the Boolean Algebra of bipartitions (Definition 293) and conjunction is just bipartition disjunction (Theorem 238).

Two different conjunctions of the voters must be different bipartitions; they have different members (Theorem 321) and so there are different voters with arrows to the two conjunctions (Definition 300) and so they must be different.

□

Theorem 325. *Let B and D be objects in B_n*

$\Omega(B \vee^b D) = \Omega(B) \cup \Omega(D)$

Proof. A voter with integer less than or equal to n votes 'yes' in the set on the left hand side iff the voter with the corresponding number is in the representation of B as a conjunction of voters or the representation of C as a conjunction of voters (By Definition 323 and Theorem 309).

An integer less than or equal to n votes 'yes' in the set on the right hand side iff the voter with the corresponding number is in the representation of B as a conjunction of voters or the representation of C as a conjunction of voters (By Definition 323).

□

Theorem 326. *Let B and D be objects in B_n*

$\Omega(B \wedge^b D) = \Omega(B) \cap \Omega(D)$

Proof. An voter with integer less than or equal to n votes 'yes' in the set on the left hand side iff the voter with the corresponding number is in the representation of B as a conjunction of voters and the representation of C as a conjunction of voters (By Definition 323 and Theorem 310).

An integer less than or equal to n votes 'yes' in the set on the right hand side iff the voter with the corresponding number is in the representation of B as a conjunction of voters and the representation of C as a conjunction of voters (By Definition 323)

□

Theorem 327. *If D is an object of B_n.*

$\Omega(D^c) = \Omega(D)^c$

Proof. For $l \leq n$

$l \in \Omega(D)^c \iff$

$\iff l \notin \Omega(D)$

$\iff v_{n,l}$ is not in the expansion of D as a conjunction of voters (Definition 323)

$\iff v_{n,l}$ votes 'no' in D (Theorem 321)

$\iff v_{n,l}$ votes 'yes' in $D^c \iff$ (Theorem 307)

$\iff v_{n,l}$ is in the expansion of D^c as a conjunction of voters (Theorem 321)

$\iff l \in \Omega(D^c)$ (Definition 323).

□

Theorem 328. *Let I be an inverted bipartition. $v_{n,i}$ votes 'yes' in $I \iff v_{n,i}$ appears in the expression of I as a disjunction of the $v_{n,r}$.*

Proof. This is the dual of Theorem 321.

□

Definition 329. $\Theta : I_n \to 2^{\{1,\ldots,n\}}$.

Let D be an object of I_n then I define $\Theta(D)$ to be the set of all r such that v_r is in the expression of D as a disjunction

Theorem 314 and Theorem 318 tells us that this is possible.

Theorem 330. Θ *is one-to-one and onto.*

Proof. This is the dual of Theorem 324.

□

Theorem 331. *Let B and D be objects in I_n*

$\Theta(B \vee^i D) = \Theta(B) \cup \Theta(D)$

Proof. This is the dual of Theorem 325.

□

Theorem 332. *Let B and D be objects in O_n*

$\Theta(B \wedge^i D) = \Theta(B) \cap \Theta(D)$

Proof. This is the dual of Theorem 326.

□

Theorem 333. *If D is an object of I_n.*

$\Theta(D^c) = \Theta(D)^c$

Proof. This is the dual of Theorem 327.

□

5 An Isomorphism Between C_n and the Functor Category of the Category of Bipartitions

Comments 334. In this section, I will show that there is an isomorphism between C_n and the category of all functors from B_n to B_1. This will give us another way to interpret the objects of C_n as SVGs and prove that C_n contains all of the SVGs.

Definition 335. $F_{n,m}$ is the category of functors from B_n to B_m.

Definition 336. $G_{n,m}$ is the category of functors from I_n to I_m.

Definition 337. G is an object of C_n and $x_{n,G}$ is a corresponding object of $F_{n,1}$. I define it by recursion on n. Bipartitions can be of the form $Vet_{n-1}(C)$ or $Dum_{n-1}(C)$ with C in B_{n-1} (Theorem 171).

$x_{n,G}(Dum_{n-1}(C)) = x_{(n-1),Dom_n(G)}(C)$

$x_{n,G}(Vet_{n-1}(D)) = x_{(n-1),Cod_n(G)}(D)$

B_1, the codomain of $x_{n,G}$, contains two objects: $\top_1$ and $Dict_1$. C_0 contains two games: $\top_0$ and $\bot_0$. There is one object in B_0: $\top_0$. The definition of x for $n = 0$ is:

$x_{0,\top_0}(\top_0) = Dict_1$

$x_{0,\bot_0}(\top_0) = \top_1$

Definition 338. G is an object of C_n and $y_{n,G}$ is the corresponding object of $G_{n,1}$. I define it by recursion on n. Inverted Bipartitions can be of the form $Pas_n(D)$ or $Dum_n(D)$ with D in I_{n-1} (Theorem 173).

$y_{n,G}(Dum_{n-1}(D)) = y_{(n-1),Dom_n(G)}(D)$

$y_{n,G}(Pas_{n-1}(D)) = y_{(n-1),Cod_n(G)}(D)$

I_1, the codomain of $y_{n,G}$, contains two objects: $\bot_1$ and $Dict_1$. C_0 contains two games: $\top_0$ and $\bot_0$. There is one object in I_0: $\bot_0$. The definition of y for $n = 0$ is:

$y_{0,\top_0}(\bot_0) = Dict_1$

$y_{0,\bot_0}(\bot_0) = \bot_1$

Theorem 339. *$(x_{n,G}(B))^* = y_{n,G}(B^*)$ for all objects G in C_n and all objects B in B_n.*

Proof. The proof is by induction on n. When $n = 0$, we need $(x_{0,G}(B))^* = y_{0,G}(B^*)$. Since there is only one bipartition in B_0, this is equivalent to

$(x_{0,G}(\top_0))^* = y_{0,G}(\top_0^*)$. Definition 337 and Definition 338 tell us that this is true for $G = \top_0$ and $G = \bot_0$. Specifically, if $G = \top_0$

$(x_{0,G}(\top_0))^*$

$= (x_{0,\top_0}(\top_0))^*$

$= Dict_1^*$ (Definition 337)

$= Dict_1$ (Theorem 138)

$= y_{0,\top_0}(\bot_0)$ (Definition 338)

$= y_{0,\top_0}(\top_0^*)$ Theorem 72

$= y_{0,G}(\top_0^*)$

If $G = \bot_0$

$(x_{0,G}(\top_0))^*$

$= (x_{0,\bot_0}(\top_0))^*$

$= \top_0^*$ Definition 337

$= \bot_0$ Theorem 72

$= y_{0,\bot_0}(\bot_0)$ Definition 338

$= y_{0,\bot_0}(\top_0^*)$ Theorem 72

$= y_{0,G}(\top_0^*)$

Let us assume that the result is true for $n = k$.

I need to show that $(x_{k+1,G}(B))^* = y_{k+1,G}(B^*)$ for all objects G in C_{k+1} and all B in B_{k+1}

Objects of B_{k+1} can be of the form $Vet_k(C)$ or $Dum_k(C)$ for C objects of B_k (Theorem 171)

First, let us sat that $B = Vet_k(C)$

$(x_{k+1,G}(B))^* = y_{k+1,G}(B^*)$

$\iff (x_{k+1,G}(Vet_k(C)))^* = y_{k+1,G}(Vet_k(C)^*)$

$\iff (x_{k+1,G}(Vet_k(C)))^* = y_{k+1,G}(Pas_k(C^*))$ Theorem 98

$\iff (x_{k,Cod_k(G)}(C))^* = y_{k+1,G}(Pas_k(C^*))$ Definition 337

$\iff (x_{k,Cod_k(G)}(C))^* = y_{k,Cod_k(G)}(C^*)$ Definition 338

Which is true by the induction hypothesis.

Next, let us sat that $B = Dum_k(C)$

$(x_{k+1,G}(B))^* = y_{k+1,G}(B^*)$

$\iff (x_{k+1,G}(Dum_k(C)))^* = y_{k+1,G}(Dum_k(C)^*)$

$\iff (x_{k+1,G}(Dum_k(C)))^* = y_{k+1,G}(Dum_k(C^*))$ Theorem 101

$\iff (x_{k,Dom_k(G)}(C))^* = y_{k+1,Dom_k(G)}(C^*)$ Definition 337 and Definition 338

Which is true by the induction hypothesis.

□

Theorem 340. *If G is an object of C_n and C is an object of B_n then $x_{n,G}(C) = Dict_1 \iff C$ wins G.*

Proof. I will prove this by induction on n.

First let us consider the case $n = 0$.

The only bipartition: $\top_0$ does not win $\bot_0$ (Definition 234, and there is no arrow from $\top_0$ to $\bot_0$). Also $x_{0,\bot_0}(\top_0) = \top_0$ Definition 337

$\top_0$ does win $\top_0$ (Definition 234, and there is an arrow from $\top_0$ to $\top_0$). Also $x_{0,\top_0}(\top_0) = Dict_1$ Definition 337.

Let us say that the Theorem holds for $n = k$

Let G be an object of C_{k+1} and C be an object in B_{k+1}. C can be of the form $Dum_k(D)$ or $Vet_k(D)$ (Theorem 171).

First, let us assume that $C = Dum_k(D)$ for D an object of B_k.

$x_{n,G}(C) = Dict_1$

$\iff x_{n,G}(Dum_k(D)) = Dict_1$

$\iff x_{n-1,Dom_k(G)}(D) = Dict_1$ By Definition 337

$\iff$ D wins $Dom_k(G)$ By The Induction Hypothesis

$\iff Dum_k(D)$ wins G Theorem 250

$\iff C$ wins G

Let us assume that $C = Vet_k(D)$ for D an object of B_k.

$x_{n,G}(C) = Dict_1$

$\iff x_{n,G}(Vet_k(D)) = Dict_1$

$\iff x_{n-1,Cod_k(G)}(D) = Dict_1$ By Definition 337

$\iff$ D wins $Cod_k(G)$ By The Induction Hypothesis

$\iff Vet_k(D)$ wins G Theorem 251

$\iff C$ wins G

This completes the induction.

□

Theorem 341. $y_{n,G}(C) = Dict_1 \iff C$ *wins* G.

Proof. Theorem 339 and Theorem 236 show that this is the dual of Theorem 340

□

Definition 342. Following Theorem 340, I say that C *wins* $x_{n,G}$ iff $x_{n,G}(C) = Dict_1$.

Definition 343. Following Theorem 341, I say that C *wins* $y_{n,G}$ iff $y_{n,G}(C) = Dict_1$.

Theorem 344. *Fixing G and considering C as the variable, $x_{n,G}(C)$ is a functor from B_n to B_1. Since these are both order categories, this amounts to saying that $x_{n,G}()$ is order preserving.*

Proof. Let us say that C and D are objects of B_n with $C \leq^b D$.

To obtain a contradiction, let us assume that it is not the case that $x_{n,G}(C) \leq^b x_{n,G}(D)$.

In this case we would need $x_{n,G}(C) = Dict_1$ and $x_{n,G}(D) = \top_0$.

In turn, this would imply that C wins G and D does not win G (Using Theorem 340).

This contradicts Theorem 244.

□

Theorem 345. *Fixing G and considering C as the variable, $y_{n,G}(C)$ is a functor from I_n to I_1. Since these are both order categories, this amounts to saying that $x_{n,G}()$ is an order-isomorphism.*

Proof. This is the dual of Theorem 344 (Using Theorem 339 and Theorem 236)

□

Theorem 346. *Fixing C and considering G as the variable, $x_{n,G}(C)$ is a functor from C_n to B_1. Since these are both order categories, this amounts to saying that $x_{n,G}(C)$, with G as the variable, is order preserving.*

Proof. Let us say that G and H are objects of C_n with $G \leq H$.

To obtain a contradiction, let us assume that it is not the case that $x_{n,G}(C) \leq^b x_{n,H}(C)$.

In this case we would need $x_{n,G}(C) = Dict_1$ and $x_{n,H}(C) = \top_1$.

In turn, this would imply that C wins G and C does not win H (Using Theorem 340).

This contradicts Theorem 248.

□

Theorem 347. *Fixing C and considering G as the variable, $y_{n,G}(C)$ is a functor from C_n to B_1. Since these are both order categories, this amounts to saying that $y_{n,G}(C)$, with G as the variable, is order preserving.*

Proof. This is the dual of Theorem 346.

□

Definition 348. If θ is an object in $F_{n,m}$ then I define B as *D-critical for θ* iff $\theta(B) = D$ and $\theta(C) = D$ and $C \leq^b B$ imply $B = C$.

Definition 349. If θ is an object in $G_{n,m}$ then I define B as *D-critical for θ* iff $\theta(B) = D$ and $\theta(C) = D$ and $C \leq^i B$ imply $B = C$.

Definition 350. If θ is an object in $F_{n,1}$ then I define B as a *winning bipartition* for θ iff $\theta(B) = Dict_1$.

Definition 351. If θ is an object in $G_{n,1}$ then I define B as a *winning inverted bipartition* for θ iff $\theta(B) = Dict_1$.

Definition 352. If θ is an object in $F_{n,1}$ then I define B as a *minimal winning bipartition* for θ iff it is $Dict_1$-critical for θ.

Definition 353. If θ is an object in $G_{n,1}$ then I define B as a *minimal winning inverted bipartition* for θ iff it is $Dict_1$-critical for θ

Definition 354. $F_{n,m}$ is a functor category and so products are defined componentwise. If θ and ϕ are objects of $F_{n,m}$ then $(\theta \wedge^{gb} \phi)(C) = \theta(C) \wedge^b \phi(C)$.

Definition 355. $G_{n,m}$ is a functor category and so products are defined componentwise. If θ and ϕ are objects of $G_{n,m}$ then $(\theta \wedge^{gi} \phi)(C) = \theta(C) \wedge^i \phi(C)$.

Definition 356. Coproducts are defined componentwise. If θ and ϕ are objects of $F_{n,m}$ then $(\theta \vee^{gb} \phi)(C) = \theta(C) \vee^{b} \phi(C)$.

Definition 357. Coproducts are defined componentwise. If θ and ϕ are objects of $G_{n,m}$ then $(\theta \vee^{gi} \phi)(C) = \theta(C) \vee^{i} \phi(C)$.

Theorem 358. *Let θ and ϕ be objects of $F_{n,m}$*

$\theta \vee^{gb} \phi = \phi \iff \theta \wedge^{gb} \phi = \theta$

Proof. $\theta \vee^{gb} \phi = \phi$

$\iff (\theta \vee^{gb} \phi)(C) = \phi(C)$ for all C in B_n

$\iff \theta(C) \vee^{b} \phi(C) = \phi(C)$ for all C in B_n Using Definition 356

$\iff \theta(C) \wedge^{b} \phi(C) = \theta(C)$ for all C in B_n By Theorem 188

$\iff (\theta \wedge^{gb} \phi)(C) = \theta(C)$ for all C in B_n Using Definition 354

$\iff \theta \wedge^{gb} \phi = \theta$

□

Theorem 359. *Let θ and ϕ be objects of $G_{n,m}$*

$\theta \vee^{gi} \phi = \phi \iff \theta \wedge^{gi} \phi = \theta.$

Proof. This is the dual of Theorem 358.

□

Definition 360. If θ and ϕ are objects of $F_{n,m}$

If $\theta \vee^{gb} \phi = \theta$ then I say that $\phi \leq^{gb} \theta$.

By Theorem 358 this is equivalent to the condition that $\theta \wedge^{gb} \phi = \phi$.

Definition 361. If θ and ϕ are objects of $G_{n,m}$

If $\theta \vee^{gi} \phi = \theta$ then I say that $\phi \leq^{gi} \theta$.

By Theorem 359 this is equivalent to the condition that $\theta \wedge^{gi} \phi = \phi$.

Theorem 362. *For ϕ and θ objects of $F_{n,m}$, $\phi \leq^{gb} \theta \iff \phi(C) \leq^{b} \theta(C)$ for all C objects of B_n.*

Proof. $\phi \leq^{gb} \theta$

$\iff \phi \vee^{gb} \theta = \theta$ (Definition 360)

$\iff (\phi \vee^{gb} \theta)(C) = \theta(C), \forall C \in B_n$

$\iff \phi(C) \vee^{b} \theta(C) = \theta(C), \forall C \in B_n$ (Definition 356)

$\iff \phi(C) \leq^{b} \theta(C), \forall C \in B_n$ (Definition 189).

□

Theorem 363. *For ϕ and θ objects of $G_{n,m}$, $\phi \leq^{gi} \theta \iff \phi(C) \leq^{i} \theta(C)$ for all C objects of I_n.*

Proof. This is the dual of Theorem 362.

□

Theorem 364. $\leq^{gb}$ *is reflexive.*

Proof. Let ϕ be an object of $F_{n,m}$

$\phi \leq^{gb} \phi$

$\iff \phi(C) \leq^{b} \phi(C)$ for all C objects of B_n (By Theorem 362)

This is true by Theorem 193.

□

Theorem 365. $\leq^{gb}$ *is antisymmetric.*

Proof. Let θ and ϕ be objects of $F_{n,m}$

$\phi \leq^{gb} \theta$ and $\theta \leq^{gb} \phi$

$\iff \phi(C) \leq^{b} \theta(C)$ and $\theta(C) \leq^{b} \phi(C)$ for all C objects of B_n (By Theorem 362)

$\iff \phi(C) = \theta(C)$ for all C objects of B_n (Theorem 194)

$\iff \phi = \theta.$

$\square$

Theorem 366. $\leq^{gb}$ *is transitive.*

Proof. Let η, θ and ϕ be objects of $F_{n,m}$

$\eta \leq^{gb} \theta$ and $\theta \leq^{gb} \phi$ (By Theorem 362)

$\iff \eta(C) \leq^{b} \theta(C)$ and $\theta(C) \leq^{b} \phi(C)$ for all C objects of B_n

$\implies \eta(C) \leq^{b} \phi(C)$ for all C objects of B_n Theorem 195

$\iff \eta \leq^{gb} \phi$ (By Theorem 362).

$\square$

Theorem 367. $\leq^{gb}$ *is a partial order.*

Proof. $\leq^{gb}$ is reflexive (Theorem 364), antisymmetric (Theorem 365) and Transitive (Theorem 366).

$\square$

Theorem 368. $\leq^{gi}$ *is reflexive.*

Proof. This is the dual of Theorem 364.

$\square$

Theorem 369. $\leq^{gi}$ *is antisymmetric.*

Proof. This is the dual of Theorem 365.

$\square$

Theorem 370. $\leq^{gi}$ *is transitive.*

Proof. This is the dual of Theorem 366.

□

Theorem 371. *$\leq^{gi}$ is a partial order.*

Proof. $\leq^{gi}$ is reflexive (Theorem 368), antisymmetric (Theorem 369) and Transitive (Theorem 370).

□

Definition 372. X_n is a function from C_n to $F_{n,1}$ such that $X_n(G) = x_{n,G}$.

Definition 373. Y_n is a function from C_n to $G_{n,1}$ such that $Y_n(G) = y_{n,G}$.

Theorem 374. *If B is a bipartition in B_n then B is the only minimal winning bipartition of $X_n(B)$.*

Proof. The proof is by induction. In $F_{0,1}$ there are two objects: $X_0(\top_0) = x_{0,\top_0}$ maps $\top_0$ to $Dict_1$ and $X_0(\bot_0) = x_{0,\bot_0}$ maps $\top_0$ to $\top_0$.

$\top_0$ is the only bipartition of B_0 and $\top_0$ wins $x_{0,\top_0}$. Also it is a MWB because (Definition 352) it is the only bipartition.

Let us assume that the theorem is true for $n = k$.

Let B be a bipartition in C_{k+1}.

B could be of the form $Vet_{k+1}(C)$ or $Dum_{k+1}(C)$ (Theorem 171)

Let us assume that $B = Vet_{k+1}(C)$.

$x_{k+1,Vet_{k+1}(C)}(Vet_{k+1}(C))$

$= x_{k,Cod_{k+2}(Vet_{k+1}(C))}(C)$ (Definition 337)

$= x_{k,C}(C)$ Definition 88 and Definition 94

$= Dict_1$ By the induction hypothesis

Let us assume that $B = Dum_{k+1}(C)$.

$x_{k+1,Dum_{k+1}(C)}(Dum_{k+1}(C))$

$= x_{k,Dom_{k+2}(Dum_{k+1}(C))}(C)$ Definition 337

$= x_{k,C}(C)$ Definition 83 and Definition 92

$= Dict_1$ By the induction hypothesis

In the case the $B = Vet_{k+1}(C)$ let us say that we have $E \leq^b B$ with $x_{k,B}(E) = Dict_1$

E could be $Vet_{k+1}(F)$ or $Dum_{k+1}(F)$ (Theorem 171 and Theorem 190).

If $E = Vet_{k+1}(F)$

E wins $x_{k+1,Vet_{k+1}(C)}$

$\iff x_{k+1,Vet_{k+1}(C)}(Vet_{k+1}(F)) = Dict_1$ (Definition 342)

$\iff x_{k,Cod_{k+2}(Vet_{k+1}(C))}(F) = Dict_1$ (Definition 337)

$\iff x_{k,C}(F) = Dict_1$ (Definition 94 and Definition 88)

$\iff F$ wins $x_{k,C}$ (Definition 342)

We know that

$E \leq^b B$

$\iff Vet_{k+1}(F) \leq^b Vet_{k+1}(C)$

$\iff F \leq^b C$ (Theorem 192)

These facts and the induction hypothesis tell us that $C = F$ and $B = E$ so we have completed the induction in this case

If $E = Dum_{k+1}(F)$

E wins $x_{k+1,Vet_{k+1}(C)}$

$\iff x_{k+1,Vet_{k+1}(C)}(Dum_{k+1}(F)) = Dict_1$

$\iff x_{k,Dom_{k+2}(Vet_{k+1}(C))}(F) = Dict_1$ Definition 337

$\iff x_{k,\perp_k}(F) = Dict_1$ (Definition 92 and Definition 88)

Theorem 340 tells us that this is only possible if F wins $\perp_k$ i.e. if there is an arrow from F to $\perp_k$ (Definition 234). This is only possible if F is $\perp_k$ (Theorem 104 and Definition 66) and $\perp_k$ is not a bipartition (Definition **??**).

So $Dum_{k+1}(F)$ cannot win $x_{k+1, Vet_{k+1}(C)}$

In the case that $B = Dum_{k+1}(C)$ let us say that we have $E \leq^b B$ with $x_{k,B}(E) = Dict_1$

E must be $Dum_{k+1}(F)$ (Theorem 171 and Theorem 190).

If $E = Dum_{k+1}(F)$

E wins $x_{k+1,Dum_{k+1}(C)}$

$\iff x_{k+1,Dum_{k+1}(C)}(Dum_{k+1}(F)) = Dict_1$

$\iff x_{k,Dom_{k+2}(Dum_{k+1}(C))}(F) = Dict_1$ (Definition 337)

$\iff x_{k,C}(F) = Dict_1$ (Definition 92 and Definition 83)

$\iff F$ wins $x_{k,C}$. (Definition 342)

We know that

$E \leq^b B$

$\iff Dum(F) \leq^b Dum(C)$

$\iff F \leq^b C$. (Theorem 192)

These facts and the induction hypothesis tell us that $C = F$ and $B = E$ so we have completed the induction in this case.

□

Theorem 375. *If B is an inverted bipartition in I_n then B is the only minimal winning inverted bipartition of $Y_n(B^*)$.*

Proof. This is the dual of Theorem 374.

□

Theorem 376. *X_n is a functor. So $F \leq G \implies X_n(F) \leq^{gb} X_n(G)$*

Proof. To obtain a contradiction, let us assume that $F \leq G$ and it is not the case that $X_n(F) \leq^{gb} X_n(G)$.

If it is not true that $X_n(F) \leq^{gb} X_n(G)$ then (Using Definition 360) it is not true that $X_n(F) = X_n(F) \wedge^{gb} X_n(G)$

This is true iff (Definition 354) there is some member, C, of B_n for which it is not the case that $x_{n,F}(C) = x_{n,F}(C) \wedge^{b} x_{n,G}(C)$.

Which is true iff (Definition 189) there is some member, C, of B_n, for which it is not the case that $x_{n,F}(C) \leq^{b} x_{n,G}(C)$.

This is true iff there is some member, C, of B_n for which $x_{n,F} = \mathit{Dict}_1$ and $x_{n,G} = \top_0$.

This is true iff (Theorem 340) there is some member, C, of B_n that wins F and doesn't win G.

This is not possible since $F \leq G$ (Theorem 248)

□

Theorem 377. *X_n is a full functor. So* $X_n(F) \leq^{gb} X_n(G) \implies F \leq G$

Proof. The proof is by induction on n

There are two objects in C_0: $\top_0$ and $\perp_0$.

There are two objects in $F_{0,1}$: $X_0(\perp_0)$ and $X_0(\top_0)$.

$X_0(\perp_0)$ is $x_{0,\perp_0}()$ (Definition 372) and maps $\top_0$ to $\top_1$ (Definition 337) and $X_0(\top_0)$ which is $x_{0,\top_0}()$ (Definition 372) maps $\top_0$ to Dict_1 (Definition 337).

Definition 360 tells us that $X_0(\perp_0) \leq^{gb} X_0(\perp_0)$; $X_0(\perp_0) \leq^{gb} X_0(\top_0)$ and $X_0(\top_0) \leq^{gb} X_0(\top_0)$. This corresponds to the fact that $\perp_0 \leq \perp_0$; $\perp_0 \leq \top_0$ and $\top_0 \leq \top_0$

Let us assume that the theorem is true for $n = k$.

Let F and G be objects in C_{k+1}.

$X_{k+1}(F) \leq^{gb} X_{k+1}(G)$

$\implies x_{k+1,F}(C) \leq^{b} x_{k+1,G}(C)$ for all C in B_{k+1} (Definition 372 and Definition 360)

$\implies x_{k+1,F}(Vet_k(B)) \leq^{b} x_{k+1,G}(Vet_k(B))$ for all B in B_{k+1} and $x_{k+1,F}(Dum_k(B)) \leq^{b} x_{k+1,G}(Dum_k(B))$ for all B in B_{k+1}. (Theorem 171)

$\implies x_{k,Cod_{k+1}(F)}(B) \leq^{b} x_{k,Cod_{k+1}(G)}(B)$ for all B in B_k and $x_{k,Dom_{k+1}(F)}(B) \leq^{b} x_{k,Dom_{k+1}(G)}(B)$ for all B in B_k. (Definition 337)

$\implies X_{n-1}(Cod_{k+1}(F)) \leq^{b} X_{n-1}(Cod_{k+1}(G))$ and $X_{n-1}(Dom_{k+1}(F)) \leq^{b} X_{n-1}(Dom_{k+1}(G))$ (Definition 372 and Definition 360)

By the induction hypothesis

$Cod_{k+1}(F) \leq Cod_{k+1}(G)$ and $Dom_{k+1}(F) \leq^{b} Dom_{k+1}(G)$ Theorem 70

$F \leq G$ and we have completed the induction

□

Theorem 378. *Y_n is a functor. So $F \leq G \implies Y_n(F) \leq^{gi} Y_n(G)$*

Proof. This is the dual of Theorem 376.

□

Theorem 379. *Y_n is a full functor. So $Y_n(F) \leq^{gi} Y_n(G) \implies F \leq G$*

Proof. This is the dual of Theorem 377.

□

Theorem 380. *If G and H are objects of B_n*

$x_{n,G\wedge H} = x_{n,G} \wedge^{gb} x_{n,H}$

Proof. $x_{n,G\wedge H}(C) = Dict_1$

$\iff$ C wins $G \wedge H$ (Theorem 340)

$\iff$ C wins G and C wins H (Theorem 258)

$\iff$ $x_{n,G}(C) = Dict_1$ and $x_{n,H}(C) = Dict_1$ (Theorem 340)

$\iff$ $x_{n,G}(C) \wedge^b x_{n,H}(C) = Dict_1$ (Definition 110)

$\iff$ $(x_{n,G} \wedge^{gb} x_{n,H})(C) = Dict_1$ (Definition 354)

□

Theorem 381. *If G and H are objects of B_n*

$x_{n,G\vee H} = x_{n,G} \vee^{gb} x_{n,H}$

Proof. $x_{n,G\vee H}(C) = Dict_1$

$\iff$ C wins $G \vee H$ (Theorem 340)

$\iff$ C wins G or C wins H (Theorem 263)

$\iff$ $x_{n,G}(C) = Dict_1$ or $x_{n,H}(C) = Dict_1$ (Theorem 340)

$\iff$ $x_{n,G}(C) \vee^b x_{n,H}(C) = Dict_1$ (Definition 110)

$\iff$ $(x_{n,G} \vee^{gb} x_{n,H})(C) = Dict_1$ (Definition 354)

□

Theorem 382. *If G and H are objects of I_n*

$y_{n,G\wedge H} = y_{n,G} \wedge^{gi} y_{n,H}$

Proof. This is the dual of Theorem 380.

□

Theorem 383. $y_{n,G\vee H} = y_{n,G} \vee^{gi} y_{n,H}$

Proof. This is the dual of Theorem 381

□

Theorem 384. *$X_n : C_n \to F(n,1)$ respects $\wedge$. That is $X_n(F \wedge G) = X_n(F) \wedge^{gb} X_n(G)$*

Proof. $X_n(F \wedge G)$

$= x_{n,F\wedge G}$ (Definition 372)

$= x_{n,F} \wedge^{gb} x_{n,G}$ (Theorem 380)

$= X_n(F) \wedge^{gb} X_n(G)$ (Definition 372)

□

Theorem 385. *$X_n : C_n \to F(n,1)$ respects $\vee$. That is $X_n(F \vee G) = X_n(F) \vee^{gb} X_n(G)$*

Proof. $X_n(F \vee G)$

$= x_{n,F\vee G}$ (Definition 372)

$= x_{n,F} \vee^{gb} x_{n,G}$ (Theorem 381)

$= X_n(F) \vee^{gb} X_n(G)$ (Definition 372)

□

Theorem 386. *$Y_n : C_n \to G(n,1)$ respects $\wedge$. That is $Y_n(F \wedge G) = Y_n(F) \wedge^{gi} Y_n(G)$*

Proof. This is the dual of Theorem 384.

□

Theorem 387. *$Y_n : C_n \to G(n,1)$ respects $\vee$. That is $Y_n(F \vee G) = Y_n(F) \vee^{gi} Y_n(G)$*

Proof. This is the dual of Theorem 385.

□

Theorem 388. *Vet_n is an object of $F_{n,n+1}$*

Proof. $Vet_n : B_n \to B_{n+1}$ by Theorem 151.

$C \leq^b D \iff Vet_n(C) \leq^b Vet_n(D)$ By Theorem 192 so Vet_n is a functor.

□

Theorem 389. *Pas_n is an object of $G_{n,n+1}$*

Proof. This is the dual of Theorem 388

□

Theorem 390. *Dum_n is an object of $F_{n,n+1}$*

Proof. $Dum_n : B_n \to B_{n+1}$ Theorem 158

$C \leq^b D \iff Dum_n(C) \leq^b Dum_n(D)$ Theorem 192 and do Dum_n is a functor.

□

Theorem 391. *Dum_n is an object of $G_{n,n+1}$*

Proof. This is the dual of Theorem 390.

□

Theorem 392. *The functor $X_n : C_n \to F_{n,1}$ is onto.*

Proof. The proof is by induction.

When $n = 0$, C_0 contains two objects: $\top_0$ and $\bot_0$. $F_{0,1}$ contains two objects one functor, $X_0(\bot_0)$, that maps $\top_0$ to $\top_1$ and another, $X_0(\top_0)$ that maps $\top_0$ to $Dict_1$. (Definitions 337 and 372). So, for $n = 0$, we can see that X_n is onto.

Let us assume that X_k is onto. I need to show that X_{k+1} is also onto.

Let as assume that θ is an object of $F(k+1,1)$. θ is a mapping from B_{k+1} to B_1. The objects of B_{k+1} are of the form $Vet_{k+1}(B)$ or $Dum_{k+1}(B)$ for B a member of B_k (Theorem 171).

Vet_k is an object of $F_{k,k+1}$ (Theorem 388) and so $\theta \circ Vet_k$ is an object of $F_{k,1}$. The induction hypothesis tells us that there is a G in C_k such that $X_k(G) = \theta \circ Vet_k$.

Dum_k is an object of $F_{k,k+1}$ (Theorem 390) and so $\theta \circ Dum_k$ is an object of $F_{k,1}$. The induction hypothesis tells us that there is an H in C_k such that $X_k(H) = \theta \circ Dum_k$.

Theorem 362 and Theorem 197 tell us that $Dum_k \leq^{gb} Vet_k$. θ is monotonic and so $\theta \circ Dum_k \leq^{gb} \theta \circ Vet_k$. By Theorem 377, $\theta \circ Dum_k \leq^{gb} \theta \circ Vet_k \implies X_k(H) \leq^{b} X_k(G) \implies H \leq G$. This fact then allows us to form an object, J, of C_{k+1} with $Dom_{k+1}(J) = H$ and $Cod_{k+1}(J) = G$. Definition 337 and Definition 372 now tell us that $X_{k+1}(J) = \theta$. Let us test $X_{k+1}(J)$ to show that it works. $X_{k+1}(J)(B) = x_{k+1,J}(B)$ (Definition 372).

Let B be an arbitrary object of B_{k+1}.

If $B = Dum_{k+1}(D)$ then

$x_{k+1,J}(B) = Dict_1$

$\iff x_{k+1,J}(Dum_{k+1}(D)) = Dict_1$

$\iff x_{k,Dom_{k+1}(J)}(D) = Dict_1$ (Definition 337)

$\iff x_{k,H}(D) = Dict_1$

$\iff \theta \circ Dum_{k+1}(D) = Dict_1$

$\theta(B) = Dict_1$.

If $B = Vet_{k+1}(D)$ then

$x_{k+1,J}(B) = Dict_1 \iff$

$x_{k+1,J}(Vet_{k+1}(D)) = Dict_1 \iff$ Definition 337

$x_{k,Cod_{k+1}(J)}(D) = Dict_1 \iff$

$x_{k,G}(D) = Dict_1 \iff$

$\theta \circ Vet_{k+1}(D) = Dict_1 \iff$

$\theta(B) = Dict_1$.

Either way, $X_{k+1}(J)(B) = x_{k+1,J}(B) = \theta(B)$. And the induction is complete.

□

Theorem 393. *The functor $Y_n : C_n \to G_{n,1}$ is onto.*

Proof. This is the dual of Theorem 392.

□

Theorem 394. *The function X_n is one-to-one.*

Proof. The proof is by induction on n.

When $n = 0$, C_0 contains two objects: $\top_0$ and $\bot_0$. $F_{0,1}$ contains two objects one functor, $X_0(\bot_0)$, that maps $\top_0$ to $\top_1$ and another, $X_0(\top_0)$ that maps $\top_0$ to $Dict_1$. (Definitions 337 and 372).

Let us assume that the theorem is true for $n = k$.

$X_{k+1}(G) = X_{k+1}(H)$

$\iff x_{k+1,G}(C) = x_{k+1,H}(C)$ for all objects C of B_{k+1} (Definition 372)

$\iff x_{k+1,G}(Vet_{k+1}(D)) = x_{k+1,H}(Vet_{k+1}(D))$ for all objects D of B_k and $x_{k+1,G}(Dum_{k+1}(D)) = x_{k+1,H}(Dum_{k+1}(D))$ for all objects D of B_k (Theorem 171)

$\iff x_{k,Cod_{k+1}(G)}(D) = x_{k,Cod_{k+1}(H)}(D)$ for all objects D of B_k and $x_{k,Dom_{k+1}(G)}(D) = x_{k,Dom_{k+1}(H)}(D)$ for all objects D of B_k (Definition 337).

$\iff Dom_{k+1}(G) = Dom_{k+1}(H)$ and $Cod_{k+1}(G) = Cod_{k+1}(H)$ (Using the induction hypothesis.)

$\iff G = H$ (Definitions 92 and 94)

□

Theorem 395. *The function Y_n is one-to-one.*

Proof. This is the dual of Theorem 395.

□

Comments 396. So we have show that X_n and Y_n are an isomorphisms of (C_n and $F_{n,1}$) and (C_n and $G_{n,1}$) respectively as categories. We have a one-to-one, onto functor between them. This gives us a new way of interpreting the C_n as simple voting games.

6 A Category Whose Objects are the C_i - L

Definition 397. Let **L** be the category that has the C_n as objects for all non-negative n and arrows that are functors from C_i to C_j that preserve $\wedge$, $\vee$, $\top_n$ and $\perp_n$.

So, if $\theta : C_n \to C_m$ is an arrow of **L** and A and B are objects of C_n, $\theta(A \wedge B) = \theta(A) \wedge \theta(B)$, $\theta(A \vee B) = \theta(A) \vee \theta(B)$, $\theta(\top_n) = \top_m$ and $\theta(\perp_n) = \perp_m$.

Theorem 398. *$Vet_n : C_n \to C_{n+1}$ preserves $\vee$, $\wedge$ and $\perp_n$ but not $\top_n$; Vet_n is not (quite) a member of* **L**.

Proof. $Vet_n : C_n \to C_{n+1}$. It has the right domain and codomain. (Definition 88 and Definition 397) .

I will now show that it preserves $\wedge$ and $\vee$.

$Vet_n(A) \wedge Vet_n(B)$

$= [\perp_{n-1}, A] \wedge [\perp_{n-1}, B]$ (Definition 88)

$= [\perp_{n-1} \wedge \perp_{n-1}, A \wedge B]$ (Definition 111).

$= [\perp_n, A \wedge B]$ (Definition 85)

$= Vet_n(A \wedge B)$ (Definition 88)

$Vet_n(A) \vee Vet_n(B)$

$= [\bot_{n-1}, A] \vee [\bot_{n-1}, B]$ (Definition 88)

$= [\bot_{n-1} \vee \bot_{n-1}, A \wedge B]$ (Definition 113).

$= [\bot_n, A \vee B]$ (Definition 85)

$= Vet_n(A \vee B)$ (Definition 88)

$Vet_n(\bot_n)$

$= [\bot_n, \bot_n]$ (Definition 88)

$= \bot_{n+1}$ (Definition 85)

Of course, *Vet* is not an arrow of **L** because

$Vet_n(\top_n)$

$= [\bot_n, \top_n]$ (Definition 88)

$= Dict_{n+1}$ (Definition 110)

$Dict_{n+1} \neq \top_{n+1}$

□

Theorem 399. *$Pas_n : C_n \to C_{n+1}$ preserves $\vee$, $\wedge$ and $\top_n$ but not $\bot_n$; Pas_n is not (quite) a member of* **L**.

Proof. This is the dual of Theorem 398.

□

Theorem 400. *$Dum_n : C_n \to C_{n+1}$ is an arrow of* **L**.

Proof. $Dum_n : C_n \to C_{n+1}$. It has the right domain and codomain. (Definition 83).

I will now show that it preserves $\wedge$ and $\vee$.

$Dum_n(A \wedge B)$

$= [A \wedge B, A \wedge B]$ (Definition 83)

$= [A, A] \wedge [B, B]$ (Definition 111)

$= Dum_n(A) \wedge Dum_n(B)$ (Definition 83)

The facts that Dum_n preserves $\vee$ and $\top_n$ are the dual results and follow directly from Definition 85

$\square$

Theorem 401. *$Dom_n : C_n \to C_{n-1}$ is an arrow of* $\mathbf{L}$.

Proof. $Dom_n : C_n \to C_{n-1}$ is it has the right domain and codomain. (Definition 92).

I will now show that it preserves $\wedge$ and $\vee$.

$Dom_n(A) \wedge Dom_n(B)$

$= Dom_n(A \wedge B)$ (Definition 111)

$Dom_n(A) \vee Dom_n(B)$

$= Dom_n(A \vee B)$ (Definition 113)

$Dom_n(\top_n)$

$= \top_{n-1}$ (Definition 92 and Definition 85)

$Dom_n(\bot_n)$

$= \bot_{n-1}$ (Definition 92 and Definition 85)

$\square$

Theorem 402. *Cod_n is an arrow of* $\mathbf{L}$.

Proof. This is the dual to Theorem 401

$\square$

Theorem 403. *Arrows of* $\mathbf{L}$ *do not necessarily preserve duality.*

Proof. Dom_n is an arrow of $\mathbf{L}$ (Theorem 401)

$Dom_n(G)^* = Cod_n(G^*)$ (Theorem 100)

Since $Cod_n \neq Dom_n$ (Definitions 92 and 94)

In general $Dom_n(G)^* \neq Dom_n(G^*)$

$\square$

Theorem 404. *An arrow of* **L** *is defined by the images of the* $\{v_{n,i} : i = 1, \ldots, n\}$ *under the arrow.*

Proof. Let us say that we have an arrow, θ, of **L** that goes from C_n to C_m

Theorem 313 tells us that we can express bipartitions in C_n as a conjunction of the v_i. Arrows of **L** respect $\wedge$ (Definition 397) and so the image, under θ, of the conjunction of the v_i is the conjunction of the images of the v_i.

Theorem 276 tells us that every object of C_n can be written as a disjunction of bipartitions. Arrows of **L** respect $\vee$ (Definition 397) and so the image of a game, under θ, is equal to the disjunction of the images of the bipartitions.

$\square$

Theorem 405. **L** *has an initial element but no terminal element.*

Proof. By Theorem 404 there are $|C_n|^m$ arrows from C_m to C_n. So there is only one arrow from C_0 to any C_n and this shows that it is the initial object. But, for any C_n, every C_m for which $m \neq 0$ has many arrows to C_n.

$\square$

Theorem 406. *Any two objects of* **L** *have a coproduct.*

Proof. The coproduct of C_n and C_m is C_{n+m}

$i : C_n \to C_{n+m}$ and $j : C_m \to C_{n+m}$ are both inclusion maps where all of the voters map onto voters.

Let $s : C_n \to C_p$ and $t : C_m \to C_p$ be arrows of **L**.

We can build, a unique u from C_{n+m} to C_p.

Voters, v in the range of i are mapped to the image (under i) of the preimage (under u) (which is a voter of C_n) of v under s so $u(v) = si^{-1}(v)$.

Voters, v in the range of j are mapped to the image (under j) of the preimage (under u) (which is a voter of C_n) of v under t so $u(v) = tj^{-1}(v)$.

□

Theorem 407. **L** *does not have all products.*

Proof. For example, what could the product of C_1 and C_1 be?

The natural choice is C_1, the projections being the identity functions.

Given arrows $s : C_1 \to C_1$ and $t : C_1 \to C_1$ such that $s(v) = \top$ and $t(v) = \bot$ then we cannot choose $u(v)$ to make the product diagram commute.

□

Definition 408. A member of **L** that has codomain C_0 is called a *valuation.*

Definition 409. Given a valuation $\theta : C_n \to C_0$, I say that a voter $v_{k,i}$ *votes 'yes'* if and only if it is mapped to $\top_0$ by θ. Otherwise, it is mapped to $\bot_0$ and I say that it *votes 'no'*.

Definition 410. Given a valuation $\theta : C_n \to C_0$, I say that an object of C_n *wins under the valuation* if and only if it is mapped to $\top_0$ by θ. Otherwise, I say that it *loses under the valuation.*

Theorem 411. *$A \wedge B$ wins under θ if and only if A wins under θ and B wins under θ.*

Proof. $A \wedge B$ wins under θ

$\implies \theta$ maps $A \wedge B$ to $\top_0$ (Definition 410)

$\implies \theta$ maps A to $\top_0$ (There is an arrow from $A \wedge B$ to A and so there is an arrow from $\theta(A \wedge B)$ to $\theta(A)$. Since $\theta(A \wedge B)$ is $\top_0$ then $\theta(A)$ must be greater than or equal to $\top_0$ and so can only be $\top_0$).

A wins under θ.

The proof that B wins under θ is similar.

Now, in the other direction.

A and B win under θ

$\implies$ that A and B are mapped to $\top_0$ by θ (Definition 410)

$\theta(A \wedge B)$

$= \theta(A) \wedge \theta(B)$ (Since θ is in **L**, Definition 397)

$= \top_0 \wedge \top_0$

$= \top_0$

And so $A \wedge B$ wins under θ. (Definition 410)

□

Theorem 412. *$A \vee B$ wins under θ if and only if A wins under θ or B wins under θ.*

Proof. Without loss of generality, let us say that A wins under θ.

$\implies \theta$ maps A to $\top_0$. (By Definition 410)

$A \vee B \geq A$ (Theorem 118), and so there is an arrow from A to $A \vee B$ (Definition 66). θ is a functor and to there is an arrow from $\theta(A)$ to $\theta(A \vee B)$. And so $\theta(A \vee B) \geq \theta(A)$ (Definition 66). $\top_0$ is maximal (Theorem 105) and so

θ maps $A \vee B$ to $\top_0$.

This tells us that $A \vee B$ wins under θ.

Let us say that θ maps $A \vee B$ to $\top_0$. To obtain a contradiction, let us assume that A and B do not win under θ. This tells us that $\theta(A) = \theta(B) = \perp_0$ (By Definition 410). This would imply (Using Definition 397) that $\theta(A \vee B) = \theta(A) \vee \theta(B) = \perp_0 \vee \perp_0 = \perp_0$. This contradiction shows that A or B must win under θ.

$\square$

Definition 413. Given a division $\theta : C_n \to C_0$, I define a *minimal winning game for* θ as being a game G such that $\theta(G) = \top_0$ and $H \leq G$ and $\theta(H) = \top_0$ imply $G = H$.

Theorem 414. *Every division has exactly one minimal winning game and it is a bipartition.*

Proof. Every division maps at least one game to $\top_0$; Definition 397 and Definition 408 tells us that they all map $\top_0$ to $\top_0$.

They must all have one minimal winning game. Start with $\top_0$. Does it have any strictly smaller games that are mapped to $\top_0$? If not then we are done if so then start again with the strictly smaller game. Of course this process must terminate as C_n has only a finite number of objects.

There can only be one minimal winning game. Let us say that there are two distinct ones: A and B. Since A and B both win under θ then so does $A \wedge B$ (Theorem 411). $A \wedge B \leq A$ (Theorem 112) and so this means that $A = A \wedge B$ (Definition 413). In a similar was $B = A \wedge B$ and so $A = A \wedge B = B$.

Now, to show that this unique minimal winning game is a bipartition, we first show that it is principal. Let us say that $A = B \vee C$.

A is winning under θ and so $\theta(A) = \top_0$ (Definition 410)

$\theta(A) = \theta(B \vee C)$

$= \theta(B) \vee \theta(C)$ (By Definition 397)

So it cannot be the case that $\theta(B) = \perp_0$ and $\theta(C) = \perp_0$.

Let us assume, WLOG, that $\theta(B) = \top_0$. We know (Theorem 118) that $B \leq B \vee C = A$ and so (by Definition 413) $B = B \vee C = A$ and A is principal (Definition 139).

A cannot be $\perp_n$ as all valuations map $\perp_0$ to $\perp_n$ (They are arrows of **L**, Definition 397)

And so A is principal and not equal to $\perp_n$ and hence it is a bipartition (Definition 145)

□

Theorem 415. *Let $\theta : C_n \to C_0$ be a valuation. Let A be the minimal winning game. If G is any game, $\theta(G) = \top_0 \iff G \geq A$*

Proof. $G \geq A$

$\implies$ There is an arrow from A to G (Definition 66)

There is an arrow from $\theta(A)$ to $\theta(G)$ (θ is a functor)

$\theta(G) = \top_0$ (Since $\theta(A) = \top_0$)

Now to carry out the proof in the opposite direction

$\theta(G) = \top_0$

$\implies \theta(G) \wedge \theta(A) = \top_0$ (Since $\theta(A) = \top_0$)

$\implies \theta(G \wedge A) = \top_0$ (θ is an arrow of **L** and Definition 397)

Since $G \wedge A \leq A$ (Theorem 112)

We know that $G \wedge A = A$ because A is the minimal winning game (Definition 413)

$\implies A \leq G$ (Theorem 112)

□

Theorem 416. *Let θ be a valuation and let A be its minimal winning game. For all r, v_r votes 'yes' in A iff $\theta(v_r) = \top_0$*

Proof. v_r votes 'yes' in A

$\implies$ There is an arrow from A to v_r (Definition 300)

$\implies$ There is an arrow from $\theta(A)$ to $\theta(v_r)$ (Since θ is a functor (Definition 408 and Definition 397))

Since A is the minimal winning game, $\theta(A) = \top_0$ and so $\theta(v_r) = \top_0$

Let us say that v_r votes 'no' in A. Definition 300 implies that there is no arrow from A to v_r.

It is not the case that $A \leq v_r$ (Definition 66)

And so Theorem 415 tells us that it is not the case that $\theta(v_r) = \top_0$.

□

Theorem 417. *For every bipartition A, there is a valuation $\theta_A : C_n \to C_0$ defined as follows:*

$\theta_A(G) = \top_0 \iff G \geq A$

$\theta_A(G) = \perp_0$ *otherwise.*

Proof. First, I need to show that θ_A is a functor.

Let us consider an arrow from G to H.

Case 1: $G \geq A$

In this case, there is an arrow from A to G (Definition 66) and $\theta_A(G) = \top_0$ (Theorem 415).

Combining the two arrows, we have an arrow from A to H. This tells us (Definition 66) that $A \leq H$ and so $\theta_A(H) = \top_0$ (Theorem 415). Therefore, under the functor, the arrow from G to H maps to an arrow from $\top_0$ to $\top_0$ (Theorem 103).

Case 2: It is not the case that $G \geq A$.

Here, if $H \geq A$ then the functor maps the arrow from G to H to the unique arrow from $\perp_0$ to $\top_0$ (Theorem 103)

If it is not the case that $H \geq A$ then the functor maps the arrow from G to H to the identity arrow from $\perp_0$ to $\perp_0$ (Theorem 102)

This functor clearly preserves $\top_0$ and $\perp_0$: $\top_0 \geq A$ for all A (Theorem 105) and so (Theorem 415) $\theta(\top_0) = \top_0$

$\perp_0 \geq A \implies A = \perp_0$ (Theorem 104) but A cannot be equal to $\perp_0$ as it is a bipartition (Definition 145) and so $\theta(\perp_0) \neq \top_0$ (By Theorem 415). Hence $\theta(\perp_0) = \perp_0$.

I need to show that $\theta_A(B \vee C)$ is equal to $\theta_A(B) \vee \theta_A(C)$.

$\theta_A(B \vee C) = \top_0$

$\iff B \vee C \geq A$ (By definition of θ_A)

$\iff B \geq A$ or $C \geq A$ (Theorem 118)

$\iff \theta_A(B) = \top_0$ or $\theta_A(C) = \top_0$) (By definition of θ_A)

$\iff \theta_A(B) \vee \theta_A(C) = \top_0$ (Theorem 118)

I need to show that $\theta_A(B \wedge C)$ is equal to $\theta_A(B) \wedge \theta_A(C)$.

$\theta_A(B \wedge C) = \top_0$

$\iff B \wedge C \geq A$ (By Definition)

$\iff B \geq A$ and $C \geq A$ (Theorem 112)

$\iff \theta_A(B) = \top_0$ and $\theta_A(C) = \top_0$ (By Definition)

$\iff \theta_A(B) \wedge \theta_A(C) = \top_0$ (Theorem 112)

$\square$

Theorem 418. *The minimal winning game of* θ_A *is* A.

Proof. C_n is a category and so there is an arrow from A to A. This and Definition 66 tell us that $A \leq A$. Theorem 415 then tells us that $\theta_A(A) = \top_0$.

Let us say that we have $B \leq A$ with $\theta_A(B) = \top_0$ then Theorem 417 would tell us that $A \leq B$ and Theorem 81 tells us that $A = B$ and so A is the minimal winning game.

□

Theorem 419. *A game* G *of* C_n *wins under* θ_A *(Definition 410)* $\iff$ A *wins* G *(Definition 234)*

Proof. G wins under θ_A

$\iff$ $G \geq A$ (Theorem 417)

$\iff$ There is an arrow from A to G (Definition 66)

$\iff$ A wins G (Definition 234)

Comments 420. There are three ways that a bipartition can win a game. The bipartition can be less than the game; its division can win the game or the bipartition can be mapped to *Dict* on the map between bipartitions that corresponds to the game. It should be quite easy to show that these are equivalent.

□

Theorem 421. *A game* G *of* C_n *wins under* θ_A *(Definition 410)* $\iff$ $(X_n(G))(A) = \mathit{Dict}_1$

Proof. Let us prove the result by induction.

In C_0 there are two objects: $\perp_0$ and $\top_0$.

There is only one valuation on C_0: $\theta_{\top_0}$, as a member of **L**, maps $\top_0$ to $\top_0$ and $\perp_0$ to $\perp_0$. (Definition 397) and (Definition 408).

$X_n(\top_0)(\top_0) = x_{n,\top_0}(\top_0)$ (Definition 372)

$x_{n,\top_0}(\top_0) = Dict_1$ (Definition 337)

$\top_0$ wins under $\theta_{\top_0}$.

$X_n(\perp_0)(\top_0) = \top_0$ (Definition 372)

$x_{n,\perp_0}(\top_0) = \top_0$ (Definition 337)

And $\perp_0$ loses under $\theta_{\top_0}$.

Let us assume that the theorem is true in C_k.

Let G be an object of C_{k+1} and A, a member of B_{k+1} and θ_A a division (a member of **L** with domain A and codomain B,).

A is a bipartition and so it must be of the form $Dum_k(B)$ or $Vet_k(B)$ (Theorem 171).

First, let us assume that it is of the form $Dum_k(B)$

G wins under θ_A

$\iff G \geq A$. (Theorem 415)

$\iff Dom_{k+1}(G) \geq B$. (Theorem 250)

$\iff Dom_{k+1}(G)$ wins under θ_B

$\iff X_k(Dom_{k+1}(G))(B) = Dict_1$ (Induction Hypothesis)

$\implies X_{k+1}(G)(Dum_k(B)) = Dict_1$ (Definition 372).

$\implies X_{k+1}(G)(A) = Dict_1$.

Next, let us assume that A is of the form $Vet_k(B)$

G wins under θ_A

$\iff G \geq A$. (Theorem 415)

$\iff Cod_{k+1}(G) \geq B$.

$\iff$ $Cod_{k+1}(G)$ wins under θ_B (Theorem 251)

$\iff$ $X_k(Cod_{k+1}(G))(B) = Dict_1$ (Induction Hypothesis)

$\implies$ $X_{k+1}(G)(Vet_k(B)) = Dict_1$ (Definition 372).

$\implies$ $X_{k+1}(G)(A) = Dict_1$

And we have completed the induction.

$\square$

Theorem 422. *Let G be an object of C_n, B a member of B_n and $\theta_B : C_n \to C_0$ be the division that has B as its bipartition.*

B wins G $\iff$ B wins $X_n(G)$

Proof. The proof is by induction on n.

For $n = 0$, there are two objects in C_0 : $\top_0$ and $\bot_0$. B_0 has one object: $\top_0$.

First, let us assume that $G = \top_0$.

$B = \top_0$ wins $G = \top_0$ there is an arrow (the identity) between them (Definition 234).

X_0 maps $\top_0$ to the functor that maps $\top_0$ to $Dict_1$ (Definition 337 and Definition 372).

$X_0(\top_0)(\top_0) = Dict_1$ and so $\top_0$ wins $X_0(\top_0)$ (Definition 350).

Second, let us assume that $G = \bot_0$.

$B = \top_0$ does not win $G = \bot_0$ there is no arrow between them (Definition 234).

X_0 maps $\bot_0$ to the functor that maps $\top_0$ to $\top_1$ (Definition 337 and Definition 372).

$X_0(\bot_0)(\top_0) = \top_1$ and so $\top_0$ does not win $X_0(\bot_0)$ (Definition 350).

Now, let us assume that we have the result for $n = k$.

Let G be an object of C_{k+1} and B be an object of B_{k+1}.

There are two cases: $B = Dum_{k+1}(D)$ and $B = Vet_{k+1}(D)$ where D is a bipartition (Theorem 171).

First, let us assume that $B = Dum_{k+1}(D)$.

Let us say that B wins G. I need to show that B wins $X_{k+1}(G)$.

There is an arrow from B to G (Definition 234).

This means that there are arrows from D to $Dom_{k+1}(G)$ and D to $Cod_{k+1}(G)$ (Theorem 70).

By the induction hypothesis, this means that D wins $X_k(Dom_k(G))$.

$\iff$ $X_k(Dom_k(G))$ maps D to $Dict_1$ (Definition 350).

$\iff$ $X_{k+1}(G)$ maps $Dum_{k+1}(D) = B$ to $Dict_1$ (Definition 337).

$\iff$ B wins $X_{k+1}(G)$ (Definition 350)

Let as say that B wins $X_{k+1}(G)$. I need to show that B wins G

$\iff$ B wins $X_{k+1}(G)$

$\iff$ $X_{k+1}(G)$ maps $Dum_{k+1}(D) = B$ to $Dict_1$ (Definition 350)

$\iff$ $X_k(Dom_k(G))$ maps D to $Dict_1$ (Definition 337)

$\iff$ D wins $X_k(Dom_k(G))$ (Definition 350)

$\iff$ D wins $Dom_k(G)$ (By the Induction Hypothesis)

$\iff$ There is an arrow from D to $Dom_k(G)$ (Definition 234

$\iff$ There are arrows from D to $Dom_k(G)$ and D to $Cod_k(G)$ (Theorem 68 and then combine the arrows)

$\iff$ There is an arrow from $B = Dum_k(D)$ to G (Theorem 70)

$\iff$ B wins G (Definition 234)

Second, let us assume that $B = Vet_{k+1}(D)$.

Let us say that B wins G. I need to show that B wins $X_{k+1}(G)$.

There is an arrow from B to G (Definition 234).

This means that there is an arrow from D to $Cod_{k+1}(G)$ (Theorem 70).

By the induction hypothesis, this means that D wins $X_k(Cod_k(G))$.

$\iff$ $X_k(Cod_k(G))$ maps D to $Dict_1$ (Definition 350).

$\iff$ $X_{k+1}(G)$ maps $Vet_{k+1}(D) = B$ to $Dict_1$ (Definition 337).

$\iff$ B wins $X_{k+1}(G)$ (Definition 350)

Let as say that B wins $X_{k+1}(G)$. I need to show that B wins G

$\iff$ B wins $X_{k+1}(G)$

$\iff$ $X_{k+1}(G)$ maps $Vet_{k+1}(D) = B$ to $Dict_1$ (Definition 350)

$\iff$ $X_k(Cod_k(G))$ maps D to $Dict_1$ (Definition 337)

$\iff$ D wins $X_k(Cod_k(G))$ (Definition 350)

$\iff$ D wins $Cod_k(G)$ (By the Induction Hypothesis)

$\iff$ There is an arrow from D to $Cod_k(G)$ (Definition 234

$\iff$ There are arrows from $\perp_k$ to $Dom_k(G)$ and D to $Cod_k(G)$ (Theorem 102)

$\iff$ There is an arrow from $B = Vet_k(D)$ to G (Theorem 70)

$\iff$ B wins G (Definition 234)

This completes the induction in both cases.

□

Comments 423. So now, we have a reasonable understanding of the divisions. What about the other arrows of **L**? We will see that they correspond to the process of forming a composite game [1, Definition 2.3.12]. The process of forming a composite game is quite a general and powerful one, particularly when we include the two degenerate games that always win and always lose. We will see that it includes the formation of products, coproducts, Boolean subgames and blocs. Isomorphisms can also be expresses using composite games.

Comments 424. Let $\theta : C_n \to C_0$ be a division and $\alpha : C_m \to C_n$ an arrow of $\mathbf{L}$. It is clear that $\theta\alpha : C_m \to C_0$ is an arrow of $\mathbf{L}$ and so it is a division.

Definition 425. Given $\alpha : C_m \to C_n$ there is an $\alpha^{'}$ that maps divisions of C_n to divisions of C_m.

$\alpha^{'}(\theta) = \theta\alpha$

Comments 426. We can also think of all of the arrows on $\mathbf{L}$ as generalisations of divisions. That is, they map all of the voters onto truth values other than just simply $\top$ and $\perp$.

Comments 427. So let G be an object of C_n and $\alpha : C_m \to C_n$ be an arrow of $\mathbf{L}$. Let us try to understand what $\alpha(G)$ is. One way to understand it, is to understand the image of $\alpha(G)$ under various divisions: θ. Of course, this is the same as the image of G under the division $\theta\alpha$.

To understand the value of G under $\theta\alpha$, first, let us work out the value of each of the v_i under $\theta\alpha$ (Theorem 404) or the image of the $\alpha(v_i)$ under θ. Then ask what the value of G would be with those values (the $\alpha(v_i)$ under θ) in place of the v_i under θ. This matches the definition of composite game given in [1, Definition 2.3.12.]. $\alpha(v_i)$ corresponds to H_i.

Such composite games can cover many operations and mappings. If α is the identity then $\alpha(v_i) = v_i \forall i$. Of course this means that $\alpha(G)$ is G (this is no surprise since α is the identity). If α maps each of the v_i onto a v_j where all of these images are different then $\alpha(G)$ is isomorphic to G as an SVG (of course, α itself need not be an isomorphism; it need not be onto).

What about if the images are not different? Then, $\alpha(G)$ is formed from G by the process of taking blocs. Divisions $\theta : C_n \to C_0$ correspond to divisions $\theta\alpha : C_m \to C_0$ in which all the voters that were mapped to the same voter in C_n go to the same truth value. If α maps each some of the v_i to themselves and others to $\top$ and $\bot$ then $\alpha(G)$ will be mapped to a Boolean subgame of G (unless all of the v_i are mapped to $\top$ and $\bot$. In this case, of course, α will just be a division and map G to $\top$ or $\bot$ - the value of the game under that division.)

7 A Notation Suggested by C_n

Comments 428. Very early on, we decided to write an arrow from A to B, an object of C_n, as $[A, B]$ (Definition 63). It is then a short jump to write objects of C_{n+1} as $[[A, B], [C, D]]$ and objects of C_{n+2} as $[[[A, B], [C, D]], [[E, F], [G, H]]]$. We can remove redundant punctuation to write these as $[ABCD]$ and $[ABCDEFGH]$. It is then a natural step to write every object of C_n as a string of the objects of C_0. On this basis, these are the objects of C_1: $[\bot\bot]$, $[\bot\top]$, $[\top\top]$. I will refer to A, B, C and D in $[ABCD]$ as its components. Given two objects (A and B) of C_1, there is an arrow from A to B (object of C_2) iff each of the two components of A are less than or equal to the corresponding component in B. This rule generalises and so given two objects (A and B) of C_n for any n there is an arrow from A to B (object of C_{n+1}) iff each of the 2^n components of A are less than or equal to the corresponding component in B. Based on this thinking, the members of C_2 are:

1. $[\bot\bot\bot\bot]$

2. $[\bot\bot\bot\top]$

3. $[\bot\bot\top\top]$

4. $[\bot\top\bot\top]$

5. $[\bot\top\top\top]$

6. $[\top\top\top\top]$

There is a natural lexicographical ordering of the objects.

Definition 429. I define a *lexicographic ordering* of the objects of C_n by recursion on n.

In $C_0, \bot \leq_L \bot, \bot \leq_L \top$ *and* $\top \leq_L \top$

In $C_n, [A, B] \leq_L [C, D] \iff A \leq_L C$ *or* $(A = C$ *and* $B \leq_L D)$

That is to say that the domain determines the lexicographic ordering (it comes first) and in the case of a tie on the domain then the codomain can adjudicate.

Definition 430. Let G be an object of C_n. I write the n^{th} component, from the right, as G_a where a is a representation of n in binary, $\top$ standing for 1 and $\bot$ standing for 0. So, for example, if G is a game with three voters, $G = [G_{\bot\bot\bot}, G_{\top\bot\bot}, G_{\bot\top\bot}, G_{\top\top\bot}, G_{\bot\bot\top}, G_{\top\bot\top}, G_{\bot\top\top}, G_{\top\top\top}]$.

I will also generalise the use of this notation. To include cases where members of the strings are games rather than voters. For example, if $G = [F, H]$. I take $G_\bot$ to mean F and $G_\top$ to mean H. If $G = [[A, B], [C, D]] = [ABCD]$ then $G_{\bot\top} = C$.

Comments 431. The idea is that G_a is the value of G when the voters are set to the string of values in a reading from left to right.

Comments 432. The subscripts, as a numbers, have a natural linear order. It is also helpful to define a partial order. This reflects the fact that they really represent bipartitions.

Definition 433. If a and b are strings, of the same length, of $\top$ and $\perp$ then we say that $a \leq_S b$ iff b has a $\top$ in every position that a has a $\top$.

Comments 434. We have an algorithm for generating the objects of C_n one by one.

1. Begin with all components equal to $\perp$.
2. Given the representation of G, a member of C_n, moving in from the right find the first $\perp$.
3. Change this to $\top$.
4. Call the subscript of this component a. If this subscript contains more than one $\top$, output the new game and go back to step two.
5. Otherwise (if the binary representation of this component's index contains just one $\top$) then carry out the following loop for each of the components to the right of the component that we have just changed.
6. Set the component to $\top$ if it has the index b such that $b >_S a$.
7. Otherwise set the component to $\perp$.
8. When all the components to the right of one with subscript a have been checked then we go back to the second step.

Comments 435. The last few steps insure monotonicity.

Comments 436. The number of games gets big quite quickly as a game is, with the constraint of monotonicity, a subset of the power set. So the number of games with n voters will roughly grow as 2^{2^n}.

Comments 437. For the C_3, the objects are:

1. $[\bot\bot\bot\bot\bot\bot\bot\bot]$
2. $[\bot\bot\bot\bot\bot\bot\bot\top]$
3. $[\bot\bot\bot\bot\bot\bot\top\top]$
4. $[\bot\bot\bot\bot\bot\top\bot\top]$
5. $[\bot\bot\bot\bot\bot\top\top\top]$
6. $[\bot\bot\bot\bot\top\top\top\top]$
7. $[\bot\bot\bot\top\bot\bot\bot\top]$
8. $[\bot\bot\bot\top\bot\bot\top\top]$
9. $[\bot\bot\bot\top\bot\top\bot\top]$
10. $[\bot\bot\bot\top\bot\top\top\top]$
11. $[\bot\bot\bot\top\top\top\top\top]$
12. $[\bot\bot\top\top\bot\bot\top\top]$
13. $[\bot\bot\top\top\bot\top\top\top]$
14. $[\bot\bot\top\top\top\top\top\top]$
15. $[\bot\top\bot\top\bot\top\bot\top]$

16. $[\bot\top\bot\top\bot\top\top\top]$

17. $[\bot\top\bot\top\top\top\top\top]$

18. $[\bot\top\top\top\bot\top\top\top]$

19. $[\bot\top\top\top\top\top\top\top]$

20. $[\top\top\top\top\top\top\top\top]$

For C_4, the objects are:

1. $[\bot\bot\bot\bot, \bot\bot\bot\bot, \bot\bot\bot\bot, \bot\bot\bot\bot]$
2. $[\bot\bot\bot\bot, \bot\bot\bot\bot, \bot\bot\bot\bot, \bot\bot\bot\top]$
3. $[\bot\bot\bot\bot, \bot\bot\bot\bot, \bot\bot\bot\bot, \bot\bot\top\top]$
4. $[\bot\bot\bot\bot, \bot\bot\bot\bot, \bot\bot\bot\bot, \bot\top\bot\top]$
5. $[\bot\bot\bot\bot, \bot\bot\bot\bot, \bot\bot\bot\bot, \bot\top\top\top]$
6. $[\bot\bot\bot\bot, \bot\bot\bot\bot, \bot\bot\bot\bot, \top\top\top\top]$
7. $[\bot\bot\bot\bot, \bot\bot\bot\bot, \bot\bot\bot\top, \bot\bot\bot\top]$
8. $[\bot\bot\bot\bot, \bot\bot\bot\bot, \bot\bot\bot\top, \bot\bot\top\top]$
9. $[\bot\bot\bot\bot, \bot\bot\bot\bot, \bot\bot\bot\top, \bot\top\bot\top]$
10. $[\bot\bot\bot\bot, \bot\bot\bot\bot, \bot\bot\bot\top, \bot\top\top\top]$
11. $[\bot\bot\bot\bot, \bot\bot\bot\bot, \bot\bot\bot\top, \top\top\top\top]$
12. $[\bot\bot\bot\bot, \bot\bot\bot\bot, \bot\bot\top\top, \bot\bot\top\top]$

13. $[\bot\bot\bot\bot, \bot\bot\bot\bot, \bot\bot\top\top, \bot\top\top\top]$

14. $[\bot\bot\bot\bot, \bot\bot\bot\bot, \bot\bot\top\top, \top\top\top\top]$

15. $[\bot\bot\bot\bot, \bot\bot\bot\bot, \bot\top\bot\top, \bot\top\bot\top]$

16. $[\bot\bot\bot\bot, \bot\bot\bot\bot, \bot\top\bot\top, \bot\top\top\top]$

17. $[\bot\bot\bot\bot, \bot\bot\bot\bot, \bot\top\bot\top, \top\top\top\top]$

18. $[\bot\bot\bot\bot, \bot\bot\bot\bot, \bot\top\top\top, \bot\top\top\top]$

19. $[\bot\bot\bot\bot, \bot\bot\bot\bot, \bot\top\top\top, \top\top\top\top]$

20. $[\bot\bot\bot\bot, \bot\bot\bot\bot, \top\top\top\top, \top\top\top\top]$

21. $[\bot\bot\bot\bot, \bot\bot\bot\top, \bot\bot\bot\bot, \bot\bot\bot\top]$

22. $[\bot\bot\bot\bot, \bot\bot\bot\top, \bot\bot\bot\bot, \bot\bot\top\top]$

23. $[\bot\bot\bot\bot, \bot\bot\bot\top, \bot\bot\bot\bot, \bot\top\bot\top]$

24. $[\bot\bot\bot\bot, \bot\bot\bot\top, \bot\bot\bot\bot, \bot\top\top\top]$

25. $[\bot\bot\bot\bot, \bot\bot\bot\top, \bot\bot\bot\bot, \top\top\top\top]$

26. $[\bot\bot\bot\bot, \bot\bot\bot\top, \bot\bot\bot\top, \bot\bot\bot\top]$

27. $[\bot\bot\bot\bot, \bot\bot\bot\top, \bot\bot\bot\top, \bot\bot\top\top]$

28. $[\bot\bot\bot\bot, \bot\bot\bot\top, \bot\bot\bot\top, \bot\top\bot\top]$

29. $[\bot\bot\bot\bot, \bot\bot\bot\top, \bot\bot\bot\top, \bot\top\top\top]$

30. $[\bot\bot\bot\bot, \bot\bot\bot\top, \bot\bot\bot\top, \top\top\top\top]$

31. $[\bot\bot\bot\bot, \bot\bot\bot\top, \bot\bot\top\top, \bot\bot\top\top]$

32. $[\bot\bot\bot\bot, \bot\bot\bot\top, \bot\bot\top\top, \bot\top\top\top]$

33. $[\bot\bot\bot\bot, \bot\bot\bot\top, \bot\bot\top\top, \top\top\top\top]$

34. $[\bot\bot\bot\bot, \bot\bot\bot\top, \bot\top\bot\top, \bot\top\bot\top]$

35. $[\bot\bot\bot\bot, \bot\bot\bot\top, \bot\top\bot\top, \bot\top\top\top]$

36. $[\bot\bot\bot\bot, \bot\bot\bot\top, \bot\top\bot\top, \top\top\top\top]$

37. $[\bot\bot\bot\bot, \bot\bot\bot\top, \bot\top\top\top, \bot\top\top\top]$

38. $[\bot\bot\bot\bot, \bot\bot\bot\top, \bot\top\top\top, \top\top\top\top]$

39. $[\bot\bot\bot\bot, \bot\bot\bot\top, \top\top\top\top, \top\top\top\top]$

40. $[\bot\bot\bot\bot, \bot\bot\top\top, \bot\bot\bot\bot, \bot\bot\top\top]$

41. $[\bot\bot\bot\bot, \bot\bot\top\top, \bot\bot\bot\bot, \bot\top\top\top]$

42. $[\bot\bot\bot\bot, \bot\bot\top\top, \bot\bot\bot\bot, \top\top\top\top]$

43. $[\bot\bot\bot\bot, \bot\bot\top\top, \bot\bot\bot\top, \bot\bot\top\top]$

44. $[\bot\bot\bot\bot, \bot\bot\top\top, \bot\bot\bot\top, \bot\top\top\top]$

45. $[\bot\bot\bot\bot, \bot\bot\top\top, \bot\bot\bot\top, \top\top\top\top]$

46. $[\bot\bot\bot\bot, \bot\bot\top\top, \bot\bot\top\top, \bot\bot\top\top]$

47. $[\bot\bot\bot\bot, \bot\bot\top\top, \bot\bot\top\top, \bot\top\top\top]$

48. $[\bot\bot\bot\bot, \bot\bot\top\top, \bot\bot\top\top, \top\top\top\top]$

49. $[\bot\bot\bot\bot, \bot\bot\top\top, \bot\top\bot\top, \bot\top\top\top]$

50. $[\bot\bot\bot\bot, \bot\bot\top\top, \bot\top\bot\top, \top\top\top\top]$

51. $[\bot\bot\bot\bot, \bot\bot\top\top, \bot\top\top\top, \bot\top\top\top]$

52. $[\bot\bot\bot\bot, \bot\bot\top\top, \bot\top\top\top, \top\top\top\top]$

53. $[\bot\bot\bot\bot, \bot\bot\top\top, \top\top\top\top, \top\top\top\top]$

54. $[\bot\bot\bot\bot, \bot\top\bot\top, \bot\bot\bot\bot, \bot\top\bot\top]$

55. $[\bot\bot\bot\bot, \bot\top\bot\top, \bot\bot\bot\bot, \bot\top\top\top]$

56. $[\bot\bot\bot\bot, \bot\top\bot\top, \bot\bot\bot\bot, \top\top\top\top]$

57. $[\bot\bot\bot\bot, \bot\top\bot\top, \bot\bot\bot\top, \bot\top\bot\top]$

58. $[\bot\bot\bot\bot, \bot\top\bot\top, \bot\bot\bot\top, \bot\top\top\top]$

59. $[\bot\bot\bot\bot, \bot\top\bot\top, \bot\bot\bot\top, \top\top\top\top]$

60. $[\bot\bot\bot\bot, \bot\top\bot\top, \bot\bot\top\top, \bot\top\top\top]$

61. $[\bot\bot\bot\bot, \bot\top\bot\top, \bot\bot\top\top, \top\top\top\top]$

62. $[\bot\bot\bot\bot, \bot\top\bot\top, \bot\top\bot\top, \bot\top\bot\top]$

63. $[\bot\bot\bot\bot, \bot\top\bot\top, \bot\top\bot\top, \bot\top\top\top]$

64. $[\bot\bot\bot\bot, \bot\top\bot\top, \bot\top\bot\top, \top\top\top\top]$

65. $[\bot\bot\bot\bot, \bot\top\bot\top, \bot\top\top\top, \bot\top\top\top]$

66. $[\bot\bot\bot\bot, \bot\top\bot\top, \bot\top\top\top, \top\top\top\top]$

67. $[\bot\bot\bot\bot, \bot\top\bot\top, \top\top\top\top, \top\top\top\top]$

68. $[\bot\bot\bot\bot, \bot\top\top\top, \bot\bot\bot\bot, \bot\top\top\top]$

69. $[\bot\bot\bot\bot, \bot\top\top\top, \bot\bot\bot\bot, \top\top\top\top]$

70. $[\bot\bot\bot\bot, \bot\top\top\top, \bot\bot\bot\top, \bot\top\top\top]$

71. $[\bot\bot\bot\bot, \bot\top\top\top, \bot\bot\bot\top, \top\top\top\top]$

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78. $[\bot\bot\bot\bot, \bot\top\top\top, \top\top\top\top, \top\top\top\top]$

79. $[\bot\bot\bot\bot, \top\top\top\top, \bot\bot\bot\bot, \top\top\top\top]$

80. $[\bot\bot\bot\bot, \top\top\top\top, \bot\bot\bot\top, \top\top\top\top]$

81. $[\bot\bot\bot\bot, \top\top\top\top, \bot\bot\top\top, \top\top\top\top]$

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126. $[\bot\bot\bot\top, \bot\top\top\top, \bot\top\bot\top, \top\top\top\top]$

127. $[\bot\bot\bot\top, \bot\top\top\top, \bot\top\top\top, \bot\top\top\top]$

128. $[\bot\bot\bot\top, \bot\top\top\top, \bot\top\top\top, \top\top\top\top]$

129. $[\bot\bot\bot\top, \bot\top\top\top, \top\top\top\top, \top\top\top\top]$

130. $[\bot\bot\bot\top, \top\top\top\top, \bot\bot\bot\top, \top\top\top\top]$

131. $[\bot\bot\bot\top, \top\top\top\top, \bot\bot\top\top, \top\top\top\top]$

132. $[\bot\bot\bot\top, \top\top\top\top, \bot\top\bot\top, \top\top\top\top]$

133. $[\bot\bot\bot\top, \top\top\top\top, \bot\top\top\top, \top\top\top\top]$

134. $[\bot\bot\bot\top, \top\top\top\top, \top\top\top\top, \top\top\top\top]$

135. $[\bot\bot\top\top, \bot\bot\top\top, \bot\bot\top\top, \bot\bot\top\top]$

136. $[\bot\bot\top\top, \bot\bot\top\top, \bot\bot\top\top, \bot\top\top\top]$

137. $[\bot\bot\top\top, \bot\bot\top\top, \bot\bot\top\top, \top\top\top\top]$

138. $[\bot\bot\top\top, \bot\bot\top\top, \bot\top\top\top, \bot\top\top\top]$

139. $[\bot\bot\top\top, \bot\bot\top\top, \bot\top\top\top, \top\top\top\top]$

140. $[\bot\bot\top\top, \bot\bot\top\top, \top\top\top\top, \top\top\top\top]$

141. $[\bot\bot\top\top, \bot\top\top\top, \bot\bot\top\top, \bot\top\top\top]$

142. $[\bot\bot\top\top, \bot\top\top\top, \bot\bot\top\top, \top\top\top\top]$

143. $[\bot\bot\top\top, \bot\top\top\top, \bot\top\top\top, \bot\top\top\top]$

144. $[\bot\bot\top\top, \bot\top\top\top, \bot\top\top\top, \top\top\top\top]$

145. $[\bot\bot\top\top, \bot\top\top\top, \top\top\top\top, \top\top\top\top]$

146. $[\bot\bot\top\top, \top\top\top\top, \bot\bot\top\top, \top\top\top\top]$

147. $[\bot\bot\top\top, \top\top\top\top, \bot\top\top\top, \top\top\top\top]$

148. $[\bot\bot\top\top, \top\top\top\top, \top\top\top\top, \top\top\top\top]$

149. $[\bot\top\bot\top, \bot\top\bot\top, \bot\top\bot\top, \bot\top\bot\top]$

150. $[\bot\top\bot\top, \bot\top\bot\top, \bot\top\bot\top, \bot\top\top\top]$

151. $[\bot\top\bot\top, \bot\top\bot\top, \bot\top\bot\top, \top\top\top\top]$

152. $[\bot\top\bot\top, \bot\top\bot\top, \bot\top\top\top, \bot\top\top\top]$

153. $[\bot\top\bot\top, \bot\top\bot\top, \bot\top\top\top, \top\top\top\top]$

154. $[\bot\top\bot\top, \bot\top\bot\top, \top\top\top\top, \top\top\top\top]$

155. $[\bot\top\bot\top, \bot\top\top\top, \bot\top\bot\top, \bot\top\top\top]$

156. $[\bot\top\bot\top, \bot\top\top\top, \bot\top\bot\top, \top\top\top\top]$

157. $[\bot\top\bot\top, \bot\top\top\top, \bot\top\top\top, \bot\top\top\top]$

158. $[\bot\top\bot\top, \bot\top\top\top, \bot\top\top\top, \top\top\top\top]$

159. $[\bot\top\bot\top, \bot\top\top\top, \top\top\top\top, \top\top\top\top]$

160. $[\bot\top\bot\top, \top\top\top\top, \bot\top\top\top, \top\top\top\top]$

161. $[\bot\top\bot\top, \top\top\top\top, \top\top\top\top, \top\top\top\top]$

162. $[\bot\top\top\top, \bot\top\top\top, \bot\top\top\top, \bot\top\top\top]$

163. $[\bot\top\top\top, \bot\top\top\top, \bot\top\top\top, \top\top\top\top]$

164. $[\bot\top\top\top, \bot\top\top\top, \top\top\top\top, \top\top\top\top]$

165. $[\bot\top\top\top, \top\top\top\top, \bot\top\top\top, \top\top\top\top]$

166. $[\bot\top\top\top, \top\top\top\top, \top\top\top\top, \top\top\top\top]$

167. $[\top\top\top\top, \top\top\top\top, \top\top\top\top, \top\top\top\top]$

Where I have put some of the commas back in to make it easier to parse the string visually (for the same reason that we write a million as $1,000,000$).

Of course, one of the drawbacks of this method of displaying objects of C_n is that the length of the string is exponential in n.

Lexicographic ordering is not the only way to arrange the objects of one of these categories. We can also draw diagrams that make the arrows between them clear.

C_0 looks like this.

$$\bot \longrightarrow \top$$

Identity arrows are not displayed. There is one for each object.

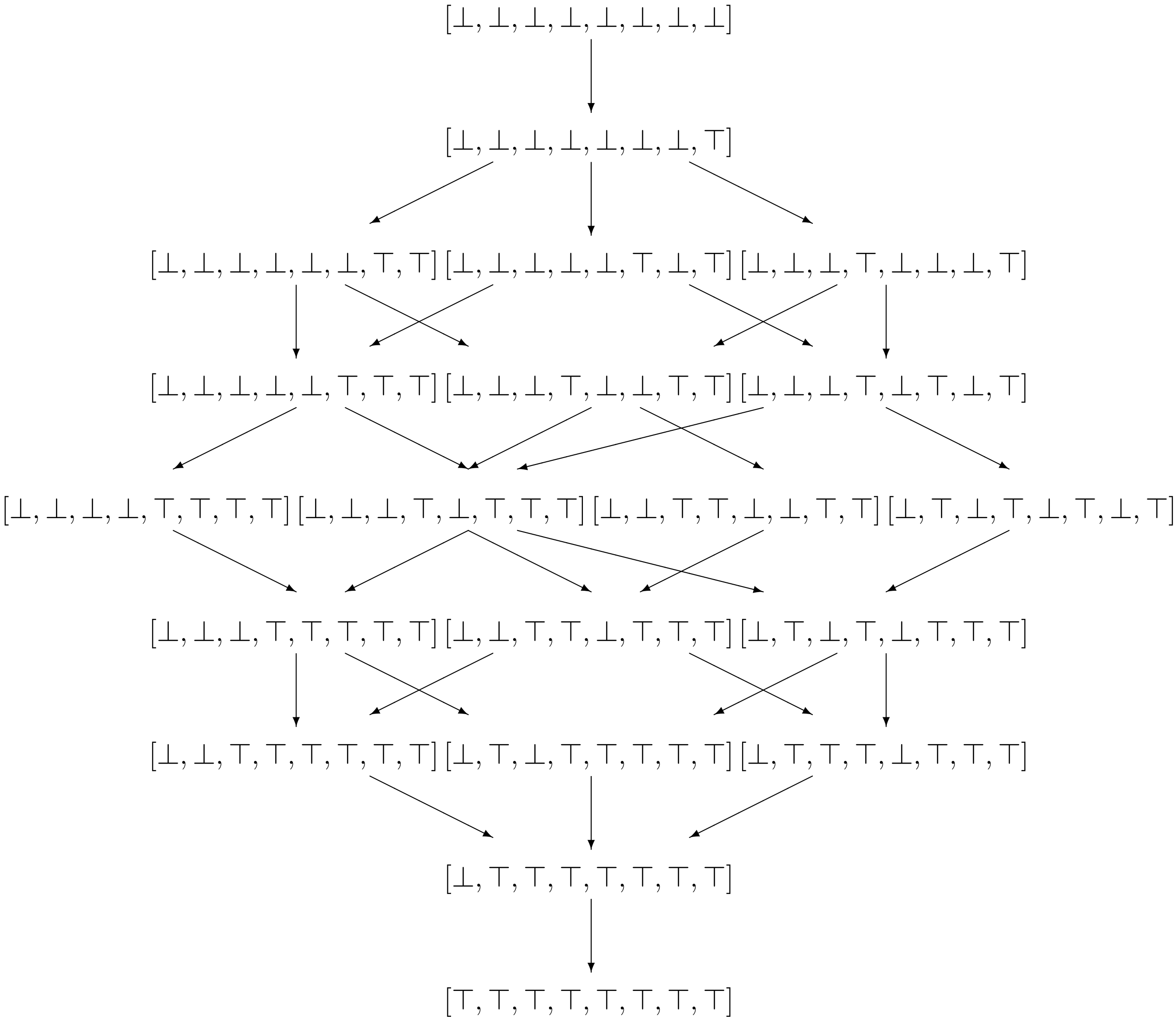
C_1 looks like this.

$$[\bot, \bot] \longrightarrow [\bot, \top] \longrightarrow [\top, \top]$$

Each of the arrows, including identities correspond to an object of C_2 which looks like this.

$$\begin{tikzcd}
& & {[\bot,\bot,\top,\top]} \arrow[dr] & & \\
{[\bot,\bot,\bot,\bot]} \arrow[r] & {[\bot,\bot,\bot,\top]} \arrow[ur] \arrow[dr] & & {[\bot,\top,\top,\top]} \arrow[r] & {[\top,\top,\top,\top]} \\
& & {[\bot,\top,\bot,\top]} \arrow[ur] & &
\end{tikzcd}$$

To display C_3, I need to rotate the picture by a right angle.



We can see that the objects are arranged in rows. The first row has the single object with no winning coalitions. The second row is the single object with one winning coalition. The third row has the three games with two winning coalitions. The r^{th} row has all of the games with r coalitions. If there are n voters then there are 2^n rows.

These four diagrams have reflectional symmetry around the line of games that have 2^{n-1} winning coalitions. Not just that, each object reflects onto its dual. For example, in the diagram of C_3, the middle object in the fourth row down is $[\bot,\bot,\bot,\top,\bot,\bot,\top,\top]$, the game with minimal winning coalitions $\{1,2\}$ and $\{2,3\}$. The object that is in the middle in the fourth row up is $[\bot,\bot,\top,\top,\bot,\top,\top,\top]$. This has minimal winning coalitions $\{2\}$ and $\{1,3\}$. These are exactly the blocking coalitions of the previous game.

It is not the case that arranging the layers with the lexicographic ordering will always lead to a game reflecting onto its dual. It is not always the case that $[A,B] \leq_L [C,D] \iff [A,B]^* \leq_L [C,D]^*$. Let us assume that this is true in C_n. Choose $[A,B]$ and $[C,D]$ in C_{n+1} with $A <_L C$ and $D <_L B$. Then $A <_L C$ tells us that $[A,B] \leq_L [C,D]$. $D <_L B$ tells us that $D^* <_L B^*$ and $[D^*,C^*] <_L [B^*,A^*]$ so $[C,D]^* <_L [A,B]^*$.

On the other hand, we can always maintain the symmetry by arranging the objects in the bottom half of the diagram to be below their duals in the top half. The arrows will also be symmetric because $G \leq H \iff H^* \leq G^*$.

If $G = [\bot\bot\bot\top\bot\top\top\top]$, I write the first half of this, $[\bot\bot\bot\top]$, as $G_\bot$ and the second half, $[\bot\top\top\top]$, as $G_\top$. In the same way, I write the second half of $G_\bot$ as $G_{\top\bot}$. So the first component of the vector is written as $G_{\bot\bot\bot}$, the second as $G_{\top\bot\bot}$, the third as $G_{\bot\top\bot}$, the fourth as $G_{\top\top\bot}$, the fifth as $G_{\bot\bot\top}$, the sixth as $G_{\top\bot\top}$, the seventh as $G_{\bot\top\top}$ and the eighth as $G_{\top\top\top}$. So, for example, in this case $G_{\top\top\bot} = \top$. We can see that $G_{\top\top\bot}$ is the value of the game if the 1^{st}, 2^{nd} and 3^{rd} voters are set to $\top$, $\top$ and $\bot$ respectively.

$\vee$ and $\wedge$ are very easy to calculate on objects written in this notation. If G and H are objects of C_n then $(G \vee H)_a = \top \iff (G_a = \top) \vee (H_a = \top)$. $(G \wedge H)_a = \top \iff (G_a = \top) \wedge (H_a = \top)$.

Comments 438. There is a strong relationship between simple voting games and logic. This was implied from the beginning by our choice to represent the possible outcomes as $\perp$ and $\top$. The reason simple voting games are called 'simple' is that the payoff function can only have two values [5] but we could have chosen 0 and 1.

First, every simple voting game can be though of as a logical proposition. The voters play the role of the prime formulae and divisions correspond to truth valuations [6, Page 20]. The divisions map each voter to $\perp$ or $\top$ and this extends to a mapping of each SVG to $\perp$ or $\top$. The logical proposition that corresponds to a game can only be built from the connectives $\wedge$ and $\vee$ because these are monotonic. If a switches from $\perp$ to $\top$ then it is impossible for $a \wedge b$ or $a \vee b$ to switch from $\top$ to $\perp$. The logical propositions that correspond to simple voting games cannot involve $\neg$ or $\rightarrow$ which are clearly not monotonic.

On the other hand, every SVG can be translated to a logical proposition using the connectives $\wedge$ and $\vee$ and in particular, this proposition can be presented in disjunctive normal form [6, Problem 6.13]. The conjunctions of voters correspond to the minimal winning coalition. Theorem 276 and Theorem 313 express this.

We can build six games with two voters:

Formula	$\perp$	$A \wedge B$	A	B	$A \vee B$	$\top$
$A = \top$ and $B = \top$	$\perp$	$\top$	$\top$	$\top$	$\top$	$\top$
$A = \top$ and $B = \perp$	$\perp$	$\perp$	$\top$	$\perp$	$\top$	$\top$
$A = \perp$ and $B = \top$	$\perp$	$\perp$	$\perp$	$\top$	$\top$	$\top$
$A = \perp$ and $B = \perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\top$

Of course, $2^{(2^n)}$ propositions can be built from n prime formulae. In this case, this is equal to sixteen.

With three voters, we have 20 simple voting games: $\top$, $A \vee B \vee C$, $A \vee B$, $A \vee C$, $B \vee C$, $A \vee (B \wedge C)$, $B \vee (A \wedge C)$, $C \vee (A \wedge B)$, A, B, C, $(A \wedge B) \vee (A \wedge C) \vee (B \wedge C)$, $(A \vee B) \wedge C$, $(A \vee C) \wedge B$, $(B \vee C) \wedge A$, $A \wedge B$, $A \wedge C$, $B \wedge C$, $A \wedge B \wedge C$ and $\perp$.

With 3 prime formulae, we have $2^{(2^3)} = 2^8 = 256$ logical propositions.

Although we cannot use negation in SVGs, we have something that looks very like it: duality.

$\neg(A \wedge B) = \neg A \vee \neg B$ and $(A \wedge B)^* = A^* \vee B^*$

$\neg\top = \perp$, $\neg\top = \perp$ and $\top^* = \perp$ and $\perp^* = \top$

For all propositions

$\neg\neg A = A$ and $(A^*)^* = A$ (Theorem 75)

But there are real differences, most obviously that, for the voters $A^* = A$ and it is obviously not the case that $\neg A = A$ for any of the prime formulae. This is where duality 'sidesteps' the lack of monotonicity in negation.

On of the consequences of this is that, for SVGs, the law of excluded middle fails; it is not always the case that $A \vee A^*$ it equal to $\top$. In fact, this is only true when A is equal to $\top$ or $\perp$. The failure of the excluded middle is shared with Intuisionistic logic but for SVGs $A \wedge A^* = \perp$ also fails in general and this holds in Intuisionistic logic.

As a matter of interest we can think about the duality operation acting on logical propositions (as SVGs) and extend it to all logical propositions. So how would we apply duality to $A \rightarrow B$? Well it's helpful to go back to what duality actually means. Instead of the winning coalitions being those that would pass the bill by voting 'yes', winning coalitions are those that could stop the bill passing by voting 'no'. It is about inserting negation in the outputs and the inputs. So $(A \wedge B)^* = \neg(\neg A \wedge \neg B) = \neg\neg A \vee \neg\neg B = A^* \vee B^*$.

On this basis, duality clearly commutes with negation. Also $(A \to B)^* = \neg(\neg A \to \neg B) = \neg(B \to A)$. So, in general, what does the dual of a proposition mean? The dual is the proposition that results if we interchange true and false in the inputs and the outputs. It is like swapping true with false entirely. In the language developed by Raymond Smullyan in [7] it is like taking a proposition from an island composed entirely of knights to an island composed entirely of knaves and changing it so that it retains its meaning on the new island.

There is another way of thinking about this. In normal propositional calculus, to every set of propositions (S), there is a corresponding Boolean algebra. The objects of this Boolean algebra are equivalence classes of formulae that are provably equivalent under the members of S and the axioms of propositional calculus. This is called the Lindenbaum algebra and is described in [6, Page 193].

The C_n are the equivalents of the Lindenbaum algebra that result from inserting propositions into S that define the behaviour of taking the dual. For example $(A \wedge B)^* = A^* \vee B^*$, $(A \vee B)^* = A^* \wedge B^*$ and $V^* = V$ for all voters. Doesn't the fact that $V = V^*$ just allow us to prove $\perp$ and hence prove any statement from any other and so the algebra collapses into a single object? No because, at the same time, we have to remove all of the axioms that define how negation works. In this case, we still get an interesting set of equivalence classes but they form a distributive lattice (Theorem 232) rather than a Boolean algebra. In this sense, the C_n are a generalisation of the idea of Lindenbaum algebra.

We can also think of the C_n as a multiple-valued logic. Each arrow of $\mathcal{L}$, $d : C_n \to C_0$ corresponds to a division of the voters. It maps each voter to $\perp$ or $\top$ and this gives us a mapping of all members of C_n to $\perp$ or $\top$. In the same way, we can think of arrows $d : C_n \to C_1$ as divisions for which the voters are allowed one of three choices ($\perp$, v or $\top$). v can play the role of abstention and this allows us to think of each of the objects of C_n as a game with abstention, as described by Felsenthal and Machover in [1, Chapter 8]. There are two problems with this, both of which we can address. In one sense C_n has too many objects and another sense it has too few. Felsenthal and Machover consider games which have abstention as an input but only have $\perp$ and $\top$ as an output. This corresponds well with real world applications. Our approach allows games that can have abstention as an outcome. Although this is not often found in the real world, it has some interest mathematically and indeed Taylor and Zwicker [3, Definition 1.1.2] consider such games. It is worth noting however that considering games with three outcomes is getting away from the concept of simple voting games as described by Shapley in [5]. There, the adjective 'simple' referred to the fact that the payoff function of the game only had two possible values. But there is another sense in which this set-up has too few games. It does not allow any monotonic game with two players that maps $\perp$ and $\top$ to abstention. The six two-player games that we have will map these to $\top$ or $\perp$ (assuming that $\vee$ and $\wedge$ keep their role as supremum and infinmum). There is a way to express every game with abstention (as output and input). In the case of C_n, we started with T which is the category with two objects $\perp$ and $\top$ and one non-identity arrow between them and then C_n was the category of functors from T to C_{n-1} (with

C_0 defined as T). To capture all of the games with abstention, we need to start with a category with three objects: $\perp$, 0 and $\top$ and arrows from $\perp$ to 0, $\perp$ to $\top$ and 0 to $\top$ (Of course this is isomorphic to C_1 but it helpful to give them different names to keep it clear that they have different meanings). If we call this Δ then we can define A_0 as Δ, A_1 as functors from Δ to Δ and A_2 as functors from Δ to A_1. C_n is the category of arrows of C_{n-1}. A_n is the category of triangles of A_{n-1}.

A_1 contains ten objects:

1. $[\perp\perp\perp]$
2. $[\perp\perp 0]$
3. $[\perp\perp\top]$
4. $[\perp 00]$
5. $[\perp 0\top]$
6. $[\perp\top\top]$
7. $[000]$
8. $[00\top]$
9. $[0\top\top]$
10. $[\top\top\top]$

A_2 contains 172 objects:

1. $[\perp\perp\perp, \perp\perp\perp, \perp\perp\perp]$

2. $[\bot\bot\bot, \bot\bot\bot, \bot\bot 0]$

3. $[\bot\bot\bot, \bot\bot\bot, \bot\bot\top]$

4. $[\bot\bot\bot, \bot\bot\bot, \bot 00]$

5. $[\bot\bot\bot, \bot\bot\bot, \bot 0\top]$

6. $[\bot\bot\bot, \bot\bot\bot, \bot\top\top]$

7. $[\bot\bot\bot, \bot\bot\bot, 000]$

8. $[\bot\bot\bot, \bot\bot\bot, 00\top]$

9. $[\bot\bot\bot, \bot\bot\bot, 0\top\top]$

10. $[\bot\bot\bot, \bot\bot\bot, \top\top\top]$

11. $[\bot\bot\bot, \bot\bot 0, \bot\bot 0]$

12. $[\bot\bot\bot, \bot\bot 0, \bot\bot\top]$

13. $[\bot\bot\bot, \bot\bot 0, \bot 00]$

14. $[\bot\bot\bot, \bot\bot 0, \bot 0\top]$

15. $[\bot\bot\bot, \bot\bot 0, \bot\top\top]$

16. $[\bot\bot\bot, \bot\bot 0, 000]$

17. $[\bot\bot\bot, \bot\bot 0, 00\top]$

18. $[\bot\bot\bot, \bot\bot 0, 0\top\top]$

19. $[\bot\bot\bot, \bot\bot 0, \top\top\top]$

20. $[\bot\bot\bot, \bot\bot\top, \bot\bot\top]$

21. $[\bot\bot\bot, \bot\bot\top, \bot 0\top]$

22. $[\bot\bot\bot, \bot\bot\top, \bot\top\top]$

23. $[\bot\bot\bot, \bot\bot\top, 00\top]$

24. $[\bot\bot\bot, \bot\bot\top, 0\top\top]$

25. $[\bot\bot\bot, \bot\bot\top, \top\top\top]$

26. $[\bot\bot\bot, \bot 00, \bot 00]$

27. $[\bot\bot\bot, \bot 00, \bot 0\top]$

28. $[\bot\bot\bot, \bot 00, \bot\top\top]$

29. $[\bot\bot\bot, \bot 00, 000]$

30. $[\bot\bot\bot, \bot 00, 00\top]$

31. $[\bot\bot\bot, \bot 00, 0\top\top]$

32. $[\bot\bot\bot, \bot 00, \top\top\top]$

33. $[\bot\bot\bot, \bot 0\top, \bot 0\top]$

34. $[\bot\bot\bot, \bot 0\top, \bot\top\top]$

35. $[\bot\bot\bot, \bot 0\top, 00\top]$

36. $[\bot\bot\bot, \bot 0\top, 0\top\top]$

37. $[\bot\bot\bot, \bot 0\top, \top\top\top]$

38. $[\bot\bot\bot, \bot\top\top, \bot\top\top]$

39. $[\bot\bot\bot, \bot\top\top, 0\top\top]$

40. $[\bot\bot\bot, \bot\top\top, \top\top\top]$

41. $[\bot\bot\bot, 000, 000]$

42. $[\bot\bot\bot, 000, 00\top]$

43. $[\bot\bot\bot, 000, 0\top\top]$

44. $[\bot\bot\bot, 000, \top\top\top]$

45. $[\bot\bot\bot, 00\top, 00\top]$

46. $[\bot\bot\bot, 00\top, 0\top\top]$

47. $[\bot\bot\bot, 00\top, \top\top\top]$

48. $[\bot\bot\bot, 0\top\top, 0\top\top]$

49. $[\bot\bot\bot, 0\top\top, \top\top\top]$

50. $[\bot\bot\bot, \top\top\top, \top\top\top]$

51. $[\bot\bot0, \bot\bot0, \bot\bot0]$

52. $[\bot\bot0, \bot\bot0, \bot\bot\top]$

53. $[\bot\bot0, \bot\bot0, \bot00]$

54. $[\bot\bot0, \bot\bot0, \bot0\top]$

55. $[\bot\bot0, \bot\bot0, \bot\top\top]$

56. $[\bot\bot 0, \bot\bot 0, 000]$

57. $[\bot\bot 0, \bot\bot 0, 00\top]$

58. $[\bot\bot 0, \bot\bot 0, 0\top\top]$

59. $[\bot\bot 0, \bot\bot 0, \top\top\top]$

60. $[\bot\bot 0, \bot\bot\top, \bot\bot\top]$

61. $[\bot\bot 0, \bot\bot\top, \bot 0\top]$

62. $[\bot\bot 0, \bot\bot\top, \bot\top\top]$

63. $[\bot\bot 0, \bot\bot\top, 00\top]$

64. $[\bot\bot 0, \bot\bot\top, 0\top\top]$

65. $[\bot\bot 0, \bot\bot\top, \top\top\top]$

66. $[\bot\bot 0, \bot 00, \bot 00]$

67. $[\bot\bot 0, \bot 00, \bot 0\top]$

68. $[\bot\bot 0, \bot 00, \bot\top\top]$

69. $[\bot\bot 0, \bot 00, 000]$

70. $[\bot\bot 0, \bot 00, 00\top]$

71. $[\bot\bot 0, \bot 00, 0\top\top]$

72. $[\bot\bot 0, \bot 00, \top\top\top]$

73. $[\bot\bot 0, \bot 0\top, \bot 0\top]$

74. $[\bot\bot 0, \bot 0\top, \bot\top\top]$

75. $[\bot\bot 0, \bot 0\top, 00\top]$

76. $[\bot\bot 0, \bot 0\top, 0\top\top]$

77. $[\bot\bot 0, \bot 0\top, \top\top\top]$

78. $[\bot\bot 0, \bot\top\top, \bot\top\top]$

79. $[\bot\bot 0, \bot\top\top, 0\top\top]$

80. $[\bot\bot 0, \bot\top\top, \top\top\top]$

81. $[\bot\bot 0, 000, 000]$

82. $[\bot\bot 0, 000, 00\top]$

83. $[\bot\bot 0, 000, 0\top\top]$

84. $[\bot\bot 0, 000, \top\top\top]$

85. $[\bot\bot 0, 00\top, 00\top]$

86. $[\bot\bot 0, 00\top, 0\top\top]$

87. $[\bot\bot 0, 00\top, \top\top\top]$

88. $[\bot\bot 0, 0\top\top, 0\top\top]$

89. $[\bot\bot 0, 0\top\top, \top\top\top]$

90. $[\bot\bot 0, \top\top\top, \top\top\top]$

91. $[\bot\bot\top, \bot\bot\top, \bot\bot\top]$

92. $[\bot\bot\top, \bot\bot\top, \bot 0\top]$

93. $[\bot\bot\top, \bot\bot\top, \bot\top\top]$

94. $[\bot\bot\top, \bot\bot\top, 00\top]$

95. $[\bot\bot\top, \bot\bot\top, 0\top\top]$

96. $[\bot\bot\top, \bot\bot\top, \top\top\top]$

97. $[\bot\bot\top, \bot 0\top, \bot 0\top]$

98. $[\bot\bot\top, \bot 0\top, \bot\top\top]$

99. $[\bot\bot\top, \bot 0\top, 00\top]$

100. $[\bot\bot\top, \bot 0\top, 0\top\top]$

101. $[\bot\bot\top, \bot 0\top, \top\top\top]$

102. $[\bot\bot\top, \bot\top\top, \bot\top\top]$

103. $[\bot\bot\top, \bot\top\top, 0\top\top]$

104. $[\bot\bot\top, \bot\top\top, \top\top\top]$

105. $[\bot\bot\top, 00\top, 00\top]$

106. $[\bot\bot\top, 00\top, 0\top\top]$

107. $[\bot\bot\top, 00\top, \top\top\top]$

108. $[\bot\bot\top, 0\top\top, 0\top\top]$

109. $[\bot\bot\top, 0\top\top, \top\top\top]$

110. $[\bot\bot\top, \top\top\top, \top\top\top]$

111. $[\bot 00, \bot 00, \bot 00]$

112. $[\bot 00, \bot 00, \bot 0\top]$

113. $[\bot 00, \bot 00, \bot\top\top]$

114. $[\bot 00, \bot 00, 000]$

115. $[\bot 00, \bot 00, 00\top]$

116. $[\bot 00, \bot 00, 0\top\top]$

117. $[\bot 00, \bot 00, \top\top\top]$

118. $[\bot 00, \bot 0\top, \bot 0\top]$

119. $[\bot 00, \bot 0\top, \bot\top\top]$

120. $[\bot 00, \bot 0\top, 00\top]$

121. $[\bot 00, \bot 0\top, 0\top\top]$

122. $[\bot 00, \bot 0\top, \top\top\top]$

123. $[\bot 00, \bot\top\top, \bot\top\top]$

124. $[\bot 00, \bot\top\top, 0\top\top]$

125. $[\bot 00, \bot\top\top, \top\top\top]$

126. $[\bot 00, 000, 000]$

127. $[\bot 00, 000, 00\top]$

128. $[\bot 00, 000, 0\top\top]$

129. $[\bot 00, 000, \top\top\top]$

130. $[\bot 00, 00\top, 00\top]$

131. $[\bot 00, 00\top, 0\top\top]$

132. $[\bot 00, 00\top, \top\top\top]$

133. $[\bot 00, 0\top\top, 0\top\top]$

134. $[\bot 00, 0\top\top, \top\top\top]$

135. $[\bot 00, \top\top\top, \top\top\top]$

136. $[\bot 0\top, \bot 0\top, \bot 0\top]$

137. $[\bot 0\top, \bot 0\top, \bot\top\top]$

138. $[\bot 0\top, \bot 0\top, 00\top]$

139. $[\bot 0\top, \bot 0\top, 0\top\top]$

140. $[\bot 0\top, \bot 0\top, \top\top\top]$

141. $[\bot 0\top, \bot\top\top, \bot\top\top]$

142. $[\bot 0\top, \bot\top\top, 0\top\top]$

143. $[\bot 0\top, \bot\top\top, \top\top\top]$

144. $[\bot 0\top, 0\top\top, 0\top\top]$

145. $[\bot 0\top, 0\top\top, \top\top\top]$

146. $[\bot 0\top, \top\top\top, \top\top\top]$

147. $[\bot\top\top, \bot\top\top, \bot\top\top]$

148. $[\bot\top\top, \bot\top\top, 0\top\top]$

149. $[\bot\top\top, \bot\top\top, \top\top\top]$

150. $[\bot\top\top, 0\top\top, 0\top\top]$

151. $[\bot\top\top, 0\top\top, \top\top\top]$

152. $[\bot\top\top, \top\top\top, \top\top\top]$

153. $[000, 000, 000]$

154. $[000, 000, 00\top]$

155. $[000, 000, 0\top\top]$

156. $[000, 000, \top\top\top]$

157. $[000, 00\top, 00\top]$

158. $[000, 00\top, 0\top\top]$

159. $[000, 00\top, \top\top\top]$

160. $[000, 0\top\top, 0\top\top]$

161. $[000, 0\top\top, \top\top\top]$

162. $[000, \top\top\top, \top\top\top]$

163. $[00\top, 00\top, 00\top]$

164. $[00\top, 00\top, 0\top\top]$

165. $[00\top, 00\top, \top\top\top]$

166. $[00\top, 0\top\top, 0\top\top]$

167. $[00\top, 0\top\top, \top\top\top]$

168. $[00\top, \top\top\top, \top\top\top]$

169. $[0\top\top, 0\top\top, 0\top\top]$

170. $[0\top\top, 0\top\top, \top\top\top]$

171. $[0\top\top, \top\top\top, \top\top\top]$

172. $[\top\top\top, \top\top\top, \top\top\top]$

A_n has a lot more members than C_n. Then again 3^{3^n} is a lot greater than 2^{2^n}

n	$\lvert C_n\rvert$	2^{2^n}	$\lvert A_n\rvert$	3^{3^n}
0	2	2	3	3
1	3	4	10	27
2	6	16	172	19683
3	20	256	?	7.6×10^{12}

One can then develop the theory of SVGs with abstention in a similar way to the theory of SVGs without. All of the A_n have a terminal and initial object (consisting entirely of $\top$ and $\perp$ respectively). Products and coproducts exist and again are equal to sup and inf (because again this category is a partially ordered set). We need three functors from A_n to A_{n-1} not just *Cod* and *Dom*. The third functor picks out the image under abstention by the final voter.

The functor that inserts a dummy from A_n to A_{n+1} is unchanged but there are six versions of the vetoer and passer. First there is a functor that maps the game to a game which is the same if the new voter votes 'no' but always passes if the new voter votes 'yes'. In this case the new voter is really a passer. In the next case the game is again the same if the new voter votes 'no' but if she votes 'yes' then she upgrades a 'no' to abstention. In the third case, the new voter is able to upgrade abstention to pass (but leaves 'no') unchanged. There are three functors, dual to this, that correspond to vetoer in the two-outcome system.

Every two-outcome game can be expressed as a disjunction of its minimal winning coalitions. When games have three outcomes, we have to think in terms of tripartitions (maps from the set of voters to the three-object partially-ordered set). We also have two types of tripartitions: minimal winning tripartitions and minimal abstaining tripartitions equivalently (and perhaps more helpfully) we can find of minimal winning tripartitions and maximal losing tripartitions.

So arrows of **L** from C_n to C_1 look a bit like games with abstention but that is not what they really are. What are they then? We saw in Comment 427 that every arrow of **L** from C_n to C_m corresponds to a composite game [1, Definition 2.3.12]. Given that C_n is defined as all the arrows from C_0 to C_{n-1} could it be that C_{n+m} corresponds to all members of **L**? The answer to this is 'no' for two reasons. First C_n was defined as all functors from C_0 to C_{n-1} not all arrows of **L** (There is only one of these from C_0 to C_n!). Second, not all games in C_{n+m} can be expressed as the composite of n games with m voters under a game with n voters. In C_3 there is only one such game (the majority game) but in other C_i there are many more. In fact, every member of C_{n+m} can be expressed as a mapping. It can be expressed as a functor from C_0^n to C_m (Theorem 394 and Theorem 392).

8 Calculating Measures of Voting Power

Definition 439. Given G in C_n, I define $|G|$ by recursion on n.

In C_0, $|\bot| = 0$ and $|\top| = 1$.

In C_n, $|[A, B]| = |A| + |B|$.

Comments 440. $|G|$ is the number of coalitions that pass the bill under G. This is also written as $\omega[G]$ in [1, Definition 3.2.8].

It is also the number of $\top$s in the representation of G as a string of $\top$s and $\bot$s (Definition 63 and Comment 428).

Theorem 441. *Let G and H be objects of C_n.*

$G \leq H \implies |G| \leq |H|$

Proof. The proof is by induction on n.

Any counterexample in C_0, would require $G = \top$ and $H = \bot$ so that $|G| = 1$ and $|H| = 0$ (Definition 439). In this case, it is not true that $|G| \leq |H|$. In this case, it is also not true that $G \leq H$ and so there is no counterexample and the implication holds.

Let us say that the theorem is true for $n = k$.

Let $[G, H]$ and $[I, J]$ be objects of C_{k+1}

$[G, H] \leq [I, J]$

$\implies G \leq I$ and $H \leq J$ (Theorem 70)

$\implies |G| \leq |I|$ and $|H| \leq |J|$ (Induction Hypothesis)

$\Longrightarrow |G| + |H| \leq |I| + |J|$ (properties of inequalities on the integers)

$\Longrightarrow |[G, H]| \leq |[I, J]|$ (Definition 439)

This completes the induction.

□

Theorem 442. *If G is an object of C_n then*

$|G^*| = 2^n - |G|$

Proof. The proof is by induction. We can see that it is true in C_0 where $|\top| = 1$ (by Definition 439), $\top^* = \perp$ (By Theorem 72) and $2^0 - 1 = |\perp| = 0$ (By Definition 439). Just to check the other case, $|\perp| = 0$ (by Definition 439), $\perp^* = \top$ (By Theorem 72) and $2^0 - 0 = |\top| = 1$ (By Definition 439).

Let us say that we have the theorem in C_n. Let $[A, B]$ be an object of C_{n+1} then.

$|[A, B]^*|$

$= |[B^*, A^*]|$ (Theorem 72)

$= |B^*| + |A^*|$ (Definition 439)

$= 2^n - |B| + 2^n - |A|$ (By the induction hypothesis)

$= 2^{n+1} - |[A, B]|$ (Definition 439)

And we have established the result by induction.

□

Theorem 443. *Given an SVG, G, the Bz measure ([1, Definition 3.2.2]) of the n^{th} voter, $\beta_n^{'}$ is equal to $(|G_\top| - |G_\perp|)/2^{n-1}$.*

Proof. We are expressing $\beta_n^{'}$ as the difference between the number of wining coalitions with the n^{th} voter votes 'yes' and the number when the n^{th} voter votes 'no'. This is consistent with [1, Theorem 3.2.4].

□

9 New Proofs of Existing Results in the Theory of Simple Voting Games

Theorem 444. *If G is an object of C_n then $G \geq G^*$ iff G is strong.*

Proof. Let us assume that $G \geq G^*$.

B does not win G

$\implies$ B^c wins G^* (Theorem 260)

$\implies$ there is an arrow from B^c to G^* (Definition 234)

$\implies$ there is an arrow from B^c to G (Combine this arrow with the arrow from G^* to G.)

$\implies$ B^c wins G

Showing that G is strong (By Definition 11)

Let us assume that G is strong so B or B^c wins G for all bipartitions B in C_n (Definition 11). I need to show that $G \geq G^*$.

By Theorem 254 we just need to show that B wins G^* implies that B wins G for all bipartitions B.

Let us say that B wins G^*. We know that B^c does not win G (Theorem 260). Since B or B^c must win G, we can deduce that B wins G and we are done.

□

Theorem 445. *If G is an object of C_n then $G \leq G^*$ iff G is proper.*

Proof. Let us assume that $G \leq G^*$.

B wins G

$\implies$ B^c does not win G^* (Theorem 260)

If B^c won G then there would be an arrow from B^c to G (Definition 234). Combining this with the arrow from G to G^* we would have an arrow from B^c to G^* (Definition 234).

Hence G is proper (Definition 10)

Let us assume that G is proper so either B or B^c loses G for all bipartitions B in C_n (Definition 10). I need to show that $G^* \geq G$.

By Theorem 254 we just need to show that B wins G implies that B wins G^* for all bipartitions B.

If B wins G then B^c cannot win G (Definition 10).

If B^c does not win G then we know that B wins G^* (Theorem 260)

Hence we have shown that $G \leq G^*$.

□

Theorem 446. *An object G, of C_n is strong $\iff$ G^* is proper.*

Proof. G is strong

$\iff$ $G \geq G^*$ Theorem 444

$\iff$ $G^* \leq (G^*)^*$ Taking the dual and using Theorem 72.

$\iff$ $G^* \leq G$ Theorem 75.

$\iff$ G^* is proper Theorem 445.

□

Theorem 447. *If G is strong then $|G| \geq 2^{n-1}$.*

Proof. If G is strong then $G \geq G^*$ (Theorem 444)

This then tells us that $|G| \geq |G^*|$ (Theorem 441).

Theorem 442 tells us that $|G^*| = 2^n - |G|$. Putting this together, we have

$|G| \geq 2^n - |G|$

$\implies 2|G| \geq 2^n$

$\implies |G| \geq 2^{n-1}$

□

Theorem 448. *If G is proper then $|G| \leq 2^{n-1}$.*

Proof. By duality.

□

Theorem 449. *An object G, of C_n is strong and proper $\iff G = G^*$*

Proof. G is strong and proper.

$\implies G \geq G^*$ and $G \leq G^*$ (Theorem 444 and Theorem 445).

$\implies G = G^*$ Theorem 81

$G = G^*$

$\implies G \geq G^*$ and $G \leq G^*$ (Definition 66 using the arrow $1_G : G \to G$)

$\implies$ that G is strong and proper. (Theorem 444 and Theorem 445)

□

Theorem 450. *[3, Proposition 1.4.9] states that if G is strong then any reduced game is strong and the reduced game is proper only if the voters in question are all dummies. It also states the dual result: if G is proper than subgames are proper and if a subgame is strong then all the relevant voters are dummies. The standard proof is about thirty lines long. With our framework, it is almost immediate.*

Proof. Let us say that $G = [A, B]$ is strong.

This means that $G^* \leq G$

$\iff [A, B]^* \leq [A, B]$

$\iff [B^*, A^*] \leq [A, B]$

so $B^* \leq A$. Since $A \leq B$ (G is monotonic). $B^* \leq B$ and B (the reduced game obtained by setting the n^{th} voter to 'yes') is strong.

If B is proper then $B \leq B^*$

Combining this with the above, we have $B \leq B^* \leq A \leq B$ so $B = B^* = A$ and the final voter is a dummy.

The proof of the other result is dual.

These results can be extended from one voter to many by an easy induction.

□

Comments 451. On [3, p27-29] Taylor and Zwicker describe the constant sum extension of a game. That is, given any game G, we can extend the voter set to build a game H which is strong and proper (constant sum) and which has G as a subgame.

In our notation, this game is written as $[G \wedge G^*, G, G^*, G \vee G^*]$. The proof that this is a game and that it is constant sum is quite complex but here it is easy to see.

We can see it is a game because $G \leq G \vee G^*$, $G^* \leq G \vee G^*$, $G \wedge G^* \leq G$ and $G \wedge G^* \leq G^*$.

A game is constant sum iff it is equal to its own dual.

$[G \wedge G^*, G, G^*, G \vee G^*]^*$

$= [[G^*, G \vee G^*]^*, [G \wedge G^*, G]^*]$

$= [(G \vee G^*)^*, G, G^*, (G \wedge G^*)^*]$

$= [G \wedge G^*, G, G^*, G \vee G^*]$

Which proves the result

Before this, Taylor and Zwicker show how to extend a strong game to a proper game and a proper game to a strong game.

Let us assume that G is strong. The extension is $[G^*, G]$. This is a game because G is strong so $G^* \leq G$. We can immediately see that $[G^*, G]$ is dual comparable because $[G^*, G]^* = [G^*, (G^*)^*] = [G^*, G]$.

If G is proper then that extension is $[G, G^*]$. This is a game because $G \leq G^*$. It is easy to see that $[G, G^*]^* = [(G^*)^*, G^*] = [G, G^*]$.

Comments 452. We want to show that the Bz score of a two-voter bloc is the sum of the scores of the two voters. [1, Theorem 3.2.18]. Using our category theoretical framework, we can do so quickly.

The power of the last two voters, in a bloc, is $(|G_{\top\top}| - |G_{\perp\perp}|)/2^{n-2}$.

The power of the final voter is $(|G_{\perp\top}| + |G_{\top\top}| - |G_{\perp\perp}| - |G_{\top\perp}|)/2^{n-1}$

The power of the penultimate voter is $(|G_{\top\perp}| + |G_{\top\top}| - |G_{\perp\perp}| - |G_{\perp\top}|)/2^{n-1}$

Adding these gives $(2|G_{\top\top}| - 2|G_{\perp\perp}|)/2^{n-1} = (|G_{\top\top}| - |G_{\perp\perp}|)/2^{n-2}$

Another way of thinking about this is to consider the following diagram.

$$\begin{tikzcd}
G_{\perp\perp} \arrow[r, "A"] \arrow[d, "C"'] \arrow[dr] & G_{\perp\top} \arrow[d, "D"'] \\
G_{\top\perp} \arrow[r, "B"] & G_{\top\top}
\end{tikzcd}$$

We can think of games in C_n as arrows of C_{n-1} or commuting squares of objects in C_{n-2} (or commuting cubes of objects C_{n-3} or n dimensional commuting cubes of objects in C_0 (that is $\perp$ and $\top$)).

The diagonal arrow represents the game that results from making a bloc of the last two voters.

The arrow A represents the dependence on the n^{th} voter, of the Boolean subgame of G in which the $(n-1)^{th}$ voter votes 'no'. B represents the dependence on the n^{th} voter, of the Boolean subgame of G in which the $(n-1)^{th}$ voter votes 'yes'. C represents the dependence on the $(n-1)^{th}$ voter, of the Boolean subgame of G in which the n^{th} voter votes 'no'. D represents the dependence on the $(n-1)^{th}$ voter, of the Boolean subgame of G in which the n^{th} voter votes 'yes'.

So the Bz score of the n^{th} voter in G is the sum of the lengths of A and B. The Bz score of the $(n-1)^{th}$ voter is the sum of the lengths of the C and D.

So the Bz count of the final voter in the game that results from bloc formation is equal to half the sum of the Bz counts of the last two voters in G.

To obtain the Bz score, we divide by 2^{n-1}. n is one smaller in the game where bloc formation has taken place so the Bz score of the bloc and the sum of the Bz scores of the other two voters are equal.

If we visualise the cube in C_{n-3} that corresponds to this game, we can see why the theorem doesn't extend to three voters. The game after bloc formation corresponds to the diagonal. For the theorem to work, we would need the sum of the counts of the three voters to be equal to four times the count of the bloc. This isn't the case in general.

10 A Bijection Function Between SVGs and Ordered Pairs of Simplicial Complexes

Definition 453. If V is a finite set then 2^V is an *abstract simplex.*

Definition 454. If 2^V is a simplex and $|V| = n$ then we say that 2^V is $(n-1)$ *dimensional.*

We admit $\{\emptyset\} = 2^\emptyset$ as the -1 dimensional simplex.

Definition 455. I define *the vertices of* 2^V to be the members of V.

Definition 456. An *(abstract) simplicial complex* is a union of a finite number of simplexes

$\mathcal{L} = \bigcup\{2^{V_1}, 2^{V_2}, 2^{V_3}, 2^{V_4}, \ldots, 2^{V_k}\}$

We admit $\emptyset$ as the empty simplicial complex. i.e. $k = 0$.

Definition 457. A *vertex of* $\mathcal{L} = \bigcup\{2^{V_1}, 2^{V_2}, 2^{V_3}, 2^{V_4}, \ldots, 2^{V_k}\}$ is a member of V_i for some i. I write the set of vertices as $V(\mathcal{L})$.

Theorem 458. *Any point v is a vertex of $\mathcal{L}$ if and only if $\{v\} \in \mathcal{L}$*

Proof. First let us say that v is a vertex of $\mathcal{L}$.

This tells us that v is a member of V and 2^V is one of the simplexes that makes up $\mathcal{L}$ (Definition 456) and Definition 457. If v is a member of V then $\{v\}$ is a member of 2^V and hence a member of $\mathcal{L}$.

If $\{v\}$ is a member of $\mathcal{L}$ then it must be a member of 2^V for some V. Which means that v must be in V for one of the 2^V that makes up the simplicial complex (Definition 456).

□

Theorem 459. *A finite set of finite sets is a simplicial complex if and only if it is closed downwards (if $B \subseteq A$ and $A \in \mathcal{L}$ then $B \in \mathcal{L}$).*

Proof. Let $\mathcal{L}$ be a simplicial complex. I need to show that it is closed downwards.

Let us say that $A \in \mathcal{L}$ then $\mathcal{L} = \cup\{2^{V_1}, 2^{V_2}, 2^{V_3}, 2^{V_4}, \ldots, 2^{V_k}\}$ (Definition 456) then A must be in 2^{V_i} for some i. If $B \subseteq A$ then it must also be the case that B is in 2^{V_i} and so B is in $\mathcal{L} = \cup\{2^{V_1}, 2^{V_2}, 2^{V_3}, 2^{V_4}, \ldots, 2^{V_k}\}$.

Now, let us start with a set that is closed downwards: $\mathcal{D}$. I need to show that it is a simplicial complex.

If it is empty then we admit it as a simplicial complex (Definition 456).

If not, then it must contain a set A. Let B be the largest set in $\mathcal{D}$ such that $A \subseteq B$. Of course, if there is nothing bigger in $\mathcal{D}$ then this will be A. Now we know that $\mathcal{D}$ contains all the subsets of B because it is closed downwards and so 2^B is part of $\mathcal{D}$. This process can be carried out for every set in $\mathcal{D}$ and so $\mathcal{D}$ must be a union of power sets.

□

Theorem 460. *Any simplicial complex can be obtained by taking a finite sets of points N and a set $\mathcal{L} \subseteq 2^N$ closed downwards. In this case $V(\mathcal{L}) \subseteq N$.*

Proof. Given a simplicial complex $\mathcal{L}$, we can chose N to be $V(\mathcal{L})$ (or some other superset of $V(\mathcal{L})$).

Then by Definition 456 and Definition 457, we can see that $\mathcal{L} \subseteq 2^N$. Theorem 459 tells us that $\mathcal{L}$ is closed downwards so $\mathcal{L}$ is the set that we need and we are done.

□

Definition 461. Given a simplicial complex $\mathcal{L}$ and a finite set N such that $V(\mathcal{L}) \subseteq N$, we put $\mathcal{L}_N^* = \{A \subseteq N : (N - A) \notin \mathcal{L}\}$. This is the *Alexander Dual of* $V(\mathcal{L})$ *relative to* N.

Theorem 462. *If* $\mathcal{L}$ *is a simplicial complex and* $V(\mathcal{L}) \subseteq N$ *then* $\mathcal{L}_N^*$ *is also a simplicial complex with* $V(\mathcal{L}_N^*) \subseteq N$

Proof. First, I will show that $\mathcal{L}_N^*$ is a simplicial complex

By Definition 461, we know that $\mathcal{L}_N^*$ is a set of sets.

By Theorem 459 all that remains is to show that $\mathcal{L}_N^*$ is closed downwards.

Given $B \in \mathcal{L}_N^*$ (of course $\mathcal{L}_N^*$ may be empty and hence closed downwards by default and a simplicial complex by Definition 456), to obtain a contradiction, let us assume that there is an $A \subseteq B$ such that $A \notin \mathcal{L}_N^*$. Definition 461 tells us that $(N - A) \in \mathcal{L}$. $B \in \mathcal{L}_N^*$ tells us that $(N - B) \notin \mathcal{L}$. $A \subseteq B$ tells us that $(N - B) \subseteq (N - A)$ this contradicts the fact that $\mathcal{L}$, as a simplicial complex, is closed downwards (Theorem 459). So, it must be the case that $A \in \mathcal{L}_N^*$ and $\mathcal{L}_N^*$ is closed downwards and is a simplcial complex.

Now I need to show that $V(\mathcal{L}_N^*) \subseteq N$

$v \in V(\mathcal{L}_N^*) \iff \{v\} \in \mathcal{L}_N^*$ (Theorem 458) and every $\{v\} \in \mathcal{L}_N^*$ is a subset of N so $v \in N$.

□

Theorem 463. *If* $\mathcal{L}$ *is a simplicial complex and* $V(\mathcal{L}) \subseteq N$ *then* $(\mathcal{L}_N^*)_N^* = \mathcal{L}$

Proof. $A \in (\mathcal{L}_N^*)_N^*$

$\iff (N - A) \notin \mathcal{L}_N^*$ (Definition 461)

$\iff (N - (N - A)) \in \mathcal{L}$ (Definition 461)

$\iff A \in \mathcal{L}$

□

Theorem 464. $V(\mathcal{L}) \subseteq V(\mathcal{L}^*_N)$ *or* $V(\mathcal{L}^*_N) \subseteq V(\mathcal{L})$

Proof. We know that $V(\mathcal{L}) \subset N$. First, let us assume that $V(\mathcal{L}) \neq N$.

Now, I need to show that $V(\mathcal{L}) \subseteq V(\mathcal{L}^*_N)$.

We can do this by showing that $\mathcal{L} \subseteq \mathcal{L}^*_N$.

Let us consider $A \in \mathcal{L}$. $(N - A)$ cannot be a member of $\mathcal{L}$. If it were then $V(\mathcal{L})$ would equal N (Definition 457) which we have assumed not to be true. Definition 461 now tells us that $A \in \mathcal{L}^*_N$ and so $V(\mathcal{L}) \subseteq V(\mathcal{L}^*_N)$

The second case is $V(\mathcal{L}^*_N) \neq N$. In this case, by duality, we know that $V(\mathcal{L}^*_N) \subseteq V(\mathcal{L})$.

In the remaining case $V(\mathcal{L}) = N$ and $V(\mathcal{L}^*_N) = N$ so $V(\mathcal{L}) = V(\mathcal{L}^*_N)$ which gives us $V(\mathcal{L}^*_N) \subseteq V(\mathcal{L})$ and $V(\mathcal{L}) \subseteq V(\mathcal{L}^*_N)$.

□

Definition 465. Let G be an SVG then I define $\mathcal{L}(G)$ to be the set of losing coalitions of G.

Theorem 466. *For every game G, $\mathcal{L}(G)$ is a simplicial complex.*

Proof. $\mathcal{L}(G)$ is a finite set of sets and so, by Theorem 459, I just need to show that it is closed downwards.

Let us say that $A \in \mathcal{L}(G)$ and $B \notin \mathcal{L}(G)$ with $B \subseteq A$. This tells us that B wins and A does not with $B \subseteq A$ this is not possible by the monotonicity of G (Definition 1).

□

Definition 467. $\theta(G) = (\mathcal{L}(G), \mathcal{L}(G^*))$

Definition 468. If G is a simple voting game then $A(G)$ is the set of voters, the assembly, of G.

Theorem 469. *If $N = A(G)$ then $\mathcal{L}(G)$ and $\mathcal{L}(G^*)$ are Alexander dual with respect to N.*

Proof. I need to prove that $\mathcal{L}(G^*) = \mathcal{L}(G)^*_N$

$S \in \mathcal{L}(G)^*_N$

$\iff$ $((N - S) \notin \mathcal{L}(G)$ (Definition 461)

$\iff$ $(N - S)$ wins G (Definition 465)

$\iff$ S does not win G^* (Definition of dual game)

$\iff$ $S \in \mathcal{L}(G^*)$ (Definition 465)

□

Theorem 470. *If G is not a dictator game then $A(G) = Max(V(\mathcal{L}(G), V(\mathcal{L}(G^*)))$. This maximum is well-defined because $V(\mathcal{L}(G)) \subseteq V(\mathcal{L}(G^*))$ or $V(\mathcal{L}(G^*)) \subseteq V(\mathcal{L}(G))$ (By Theorem 464 and Theorem 469).*

Proof. □

First, we ask if G contains a passer.

If it does then (I will call it v_1) v_1 will not be in $V(\mathcal{L}(G))$. v_1 cannot be in any sets that lose G.

However, in this case G cannot have a vetoer. It cannot be v_1 (for that would make v_1 a dictator) and it cannot be any other voter (because that would conflict with the role of v_1 as a passer).

This means that every member of $A(G)$ must be present in $V(\mathcal{L}(G^*))$. If a voter is not in $V(\mathcal{L}(G^*))$ then it must be a vetoer because that would mean that every coalition that it is in is blocking (wins the dual game).

On the other hand, if there is no passer in G then every member of $A(G)$ must be present in $V(\mathcal{L}(G))$ because if a voter was not present then every set it was in would win, making it a passer.

Theorem 471. *If G is a dictator game then $\mathcal{L}(G) = \mathcal{L}(G^*)$ equals the simplex whose vertices are the dummies of G*

Proof. Every voter that is not the dictator must clearly be in $\mathcal{L}(G)$ and $\mathcal{L}(G^*)$ - as a singleton they all lose (and use Theorem 458).

And the dictator can never be in a losing, or non-blocking set.

□

Theorem 472. *θ is one-to-one*

Proof. To obtain a contradiction, let us assume that $\theta(G) = \theta(H)$.

We know that $\mathcal{L}(G) = \mathcal{L}(H)$ and so they both have the same losing coalitions. This means that one could only differ from the other by the addition of a passer.

But also, by Theorem 471 and Theorem 470 we know that both games have the same assembly (because the two objects in the ordered pair θ define the assembly). So they cannot differ by a passer.

□

Theorem 473. *θ is onto the set of pairs of simplicial complexes: (S, T) such that $S = T^*_N$ where N is $Max(V(\mathcal{L}(G), V(\mathcal{L}(G^*)))$ as described in Theorem 470 or the underlying set of S and T, with a single voter added, when $S = T$ and they are both a simplex (Theorem 471).Of course in this case $S = T = T^*_N$.*

Proof. Let us assume that we have (S, T) as described above with an associated assembly, again described above, N.

I will show that $2^N - S$ is a simple voting game and $\theta(2^N - S) = (S, T)$. I will define H to be $2^N - S$

If $U \in H$ and $U \subseteq V$ then $U \notin S$ so $V \notin S$ (as S is closed downwards) and so $V \in H$ hence H is monotonic and an SVG.

It is clear that $\mathcal{L}(2^N - S) = S$.

$\mathcal{L}((2^N - S)^*)$

$= \mathcal{L}((2^N - S))^*_N$ (Theorem 469)

$= S^*_N$

$= T$ By assumption

□

Comments 474. So, why has this link not been discovered up until now? It is clear that the set of losing coalitions is closed downwards. Having seen this, it is only a small jump to realise that this is a simplicial complex. Of course, these simplcial complexes do not define the games, one also needs the set of losing coalitions of the dual (non-blocking coalitions) for that. Perhaps these did not occur to people because, given the game theoretic roots of SVGs, it is natural to look at the winning coalitions rather than those that lose.

Theorem 475. *An SVG G, is strong* $\iff \mathcal{L}(G) \subseteq \mathcal{L}(G^*)$

Proof. Let us assume that G is strong.

$S \in \mathcal{L}(G)$

$\implies S \notin G$

$\implies (N - S) \in G$ because G is strong (Definition 11)

$\implies S$ is not a blocking coalition for G.

$\implies S \notin G^*$ (Definition 19)

$\implies S \in \mathcal{L}(G^*)$

So $\mathcal{L}(G) \subseteq \mathcal{L}(G^*)$.

Let us assume that $\mathcal{L}(G) \subseteq \mathcal{L}(G^*)$

If S loses G then we need to show that $(N - S)$ wins G.

S loses G

$\implies (N - S)$ is blocking

$\implies (N - S) \in G^*$ (Definition 19)

$\implies (N - S) \notin \mathcal{L}(G^*)$

$\implies (N - S) \notin \mathcal{L}(G)$

$\implies (N - S) \in G$

$\implies (N - S)$ wins G

$\square$

Theorem 476. *An SVG G, is proper* $\iff \mathcal{L}(G^*) \subseteq \mathcal{L}(G)$

Proof. G is proper

$\iff G^*$ is strong (Theorem 446).

$\iff \mathcal{L}(G^*) \subseteq \mathcal{L}((G^*)^*)$ (Theorem 475)

$\iff \mathcal{L}(G^*) \subseteq \mathcal{L}(G)$ (Theorem 75).

$\square$

Theorem 477. *An SVG G, is proper and strong* $\iff \mathcal{L}(G^*) = \mathcal{L}(G)$

Proof. If G is strong and proper then $\mathcal{L}(G) \subseteq \mathcal{L}(G^*)$ and $\mathcal{L}(G^*) \subseteq \mathcal{L}(G)$). This implies $\mathcal{L}(G) = \mathcal{L}(G^*)$

If $\mathcal{L}(G) = \mathcal{L}(G^*)$ then of course $\mathcal{L}(G) \subseteq \mathcal{L}(G^*)$ and $\mathcal{L}(G^*) \subseteq \mathcal{L}(G)$) and G is strong and proper

$\square$

Comments 478. Of course, these are interesting because $\mathcal{L}(G)$ and $\mathcal{L}(G^*)$ are the two components of $\theta(G)$.

Definition 479. Given an two ordered pairs of simplicial complexes (S_1, S_2) and (T_1, T_2). I say that $(S_1, S_2) \leq (T_1, T_2) \iff (T_1 \subseteq S_1 \text{ and } S_2 \subseteq T_2)$

Theorem 480. *If G and H have the same set of voters then $G \leq H$ (i.e. H wins whenever G does) $\iff \theta(G) \leq \theta(H)$.*

Proof. Let us say that $G \leq H$.

I need to show that $\theta(G) \leq \theta(H)$ which means $\mathcal{L}(H) \subseteq \mathcal{L}(G)$ and $\mathcal{L}(G^*) \subseteq \mathcal{L}(H^*)$

Let us say that $S \in \mathcal{L}(H)$

$\implies S \notin H$

$\implies S \notin G$

$\implies S \in \mathcal{L}(G)$

So $\mathcal{L}(H) \subseteq \mathcal{L}(G)$

Let us say that $S \in \mathcal{L}(G^*)$

$\implies S \notin G^*$

$\implies (N - S) \in G$ (Definition 19)

$\implies (N - S) \in H$

$\implies S \notin H^*$ (Definition 19)

$\implies S \in \mathcal{L}(H^*)$

So $\mathcal{L}(G^*) \subseteq \mathcal{L}(H^*)$

Let us say that $\theta(G) \leq \theta(H)$. I need to show that $G \leq H$.

Let us say that S wins G.

This means that $S \notin \mathcal{L}(G)$

Hence $S \notin \mathcal{L}(H)$ $(\theta(G) \leq \theta(H) \implies \mathcal{L}(H) \subseteq \mathcal{L}(G))$

So S wins H.

□

Comments 481. So $\theta(G) \leq \theta(H)$ tells us that $G \leq H$. What about when G and H have different voters? If we know that $\theta(G) \leq \theta(H)$ is there a meaningful relationship between G and H? It turns out that the answer is 'yes'.

Theorem 482. *If H differs from G by the addition of a vetoer then $\theta(H) \leq \theta(G)$.*

If H differs from G by the addition of a passer then $\theta(G) \leq \theta(H)$.

Proof. If H differs from G by the addition of a passer then $\mathcal{L}(G) = \mathcal{L}(H)$ because a passer can never be in a losing set and all of the losing sets of H are still losing in G. $\mathcal{L}(G^*)$ is only equal to $\mathcal{L}(H^*)$ if the extra voter is also a vetoer (and hence a dictator). This could only have been the case if G was the game that always loses. On the other hand, we can say that $\mathcal{L}(G^*) \subseteq \mathcal{L}(H^*)$ because the new voter may well be involved in new non-blocking coalitions and all the old non-blocking coalitions are still non-blocking (because the new voter is simply a passer). This tells us that $\theta(G) \leq \theta(H)$.

By duality if H differs from G by the addition of a vetoer then $\mathcal{L}(G^*) = \mathcal{L}(H^*)$ and $\mathcal{L}(G) \subseteq \mathcal{L}(H)$. In this case, we can see that $\theta(H) \leq \theta(G)$.

□

Comments 483. This means that if $\theta(H) \leq \theta(G)$ and the voters of G are a superset of those of H then we can think of G as being formed by adding passers to a game that has the same number of voters as H and is at least as big as H.

if $\theta(G) \leq \theta(H)$ and the voters of G are a superset of those of H then we can think of G as being formed by adding vetoers to a game that has the same number of voters as H and is no bigger than H.

It doesn't feel unreasonable to extend $\geq$ to a relation between games that may have different numbers of voters, defining $G \leq H \iff \theta(G) \leq \theta(H)$.

This new relation means that $G \leq H$ iff the result of G under a division is always less than the result of H. If the voters of one game are a subset of another then the voters in the set-theoretic difference are just ignored when the game with fewer voters is evaluated. The point is that adding a passer can make the result nothing but better and adding a vetoer can make it nothing but worse.

Of course the partial order remains silent on the addition of other types of voters because their impact is mixed.

Comments 484. We can see that the results are coming in dual pairs. We saw the same thing in the lattice solution. We can now see that the fact that that mapping of SVGs onto simplicial complexes has a kind of symmetry, under the taking of duals, is critical to its correctness and usefulness.

Theorem 485. $\theta(A \wedge B) = (\mathcal{L}(A \wedge B), \mathcal{L}((A \wedge B)^*)) = (\mathcal{L}(A) \cup \mathcal{L}(B), \mathcal{L}(A^*) \cap \mathcal{L}(B^*))$

$\theta(A \vee B) = (\mathcal{L}(A \vee B), \mathcal{L}((A \vee B)^*)) = (\mathcal{L}(A) \cap \mathcal{L}(B), \mathcal{L}(A^*) \cup \mathcal{L}(B^*))$

Proof. $S \in \mathcal{L}(A \wedge B)$ if and only if S loses $A \wedge B$ iff S loses A or S loses B (Definition 24) iff $S \in \mathcal{L}(A)$ or $S \in \mathcal{L}(B)$ iff $S \in \mathcal{L}(A) \cup \mathcal{L}(B)$.

$S \in \mathcal{L}((A \wedge B)^*)$ if and only if S loses $(A \wedge B)^*$ iff S loses $A^* \vee B^*$ (Theorem 117) iff S loses A^* and S loses B^* (Definition 25) iff $S \in \mathcal{L}(A^*)$ and $S \in \mathcal{L}(B^*)$ iff $S \in \mathcal{L}(A^*) \cap \mathcal{L}(B^*)$.

The second result is dual.

□

Theorem 486. $\mathcal{L}(Pas_{n-1}(G)) = \mathcal{L}(G)$

$$\mathcal{L}(Pas_{n-1}(G)^*) = \mathcal{P}(\{1, \ldots, n-1\}) \cup \{S \cup \{n\} : S \in \mathcal{L}(G^*)\}.$$

Proof. $\mathcal{L}(Pas_{n-1}(G)) = \mathcal{L}(G)$ because passers are not in any losing coalitions.

Any coalition that does not include the passer cannot be blocking. So $\mathcal{P}(\{1, \ldots, n-1\})$ is contained in $\mathcal{L}(Pas_{n-1}(G))$.

n, added to a coalition that was not blocking for G, is always not blocking for $\mathcal{L}(Pas_{n-1}(G))$ as ensuring that n votes 'no' just returns G and we know that remaining voters cannot block in G.

□

Theorem 487. $\mathcal{L}(Vet_{n-1}(G)^*) = \mathcal{L}(G^*)$

$$\mathcal{L}(Vet_{n-1}(G)) = \mathcal{P}(\{1, \ldots, n-1\}) \cup \{S \cup \{n\} : S \in \mathcal{L}(G)\}.$$

Proof. This is the dual of Theorem 486.

□

Comments 488. The fact that $\mathcal{L}(Pas_{n-1}(G)) = \mathcal{L}(G)$ reminds us about the way in which $\mathcal{L}(G)$ does not specify G. The addition of a passer makes no difference to $\mathcal{L}(G)$ as it doesn't show up in any losing sets. In the same way, $\mathcal{L}(G^*)$ doesn't specify G as the addition of a vetoer makes no difference. It doesn't show up in any non-blocking coalitions.

This suggests a second way of specifying an SVG, as an ordered pair of a simplicial complex and a set of passers. This is not without prospects. The losing coalitions of $G_V \wedge H_W$ are the union of the losing coalitions of G_V and the losing coalitions of H_W. The passers of $G_V \wedge H_W$ are just the intersection of the passers of G_V and the passers of H_W. So if $V \cap W = \emptyset$ and we have a product then there are no passers in $G_V \wedge H_W$. The passers of $G_V \vee H_W$ are just the union of the passers of G_V and the passers of H_W. So this method has some merits but it doesn't quite seem to have the symmetry of the other approach. Duals are not as easy to handle and Theorem 493 and Theorem 494 would not have such power. In fact, we can also see that the fact that the passers of $G_V \wedge H_W$ are the intersection of the passers of G_V and the passers of H_W is related to the fact that $\mathcal{L}((A \wedge B)^*) = \mathcal{L}(A^*) \cap \mathcal{L}(B^*)$. If we do not have a dictator then passers cannot be vetoers so the sets containing each passer as a singleton must show up in the set of losing sets of the dual game. So these methods are not that different. To me, it seems that representing as a simplicial complex and a set of passers is a poor relation of the method that represents a game as pair of simplicial complexes.

Of course, there is a third way to represent a game: as a simplicial complex of losing coalition and a set of voters. We cannot work out the set of voters from the losing coalitions because passers do not show up. The other two methods that we have seen for representing a game both allow us to do this indirectly by working out the set of voters.

Theorem 489. *Addition of dummies.*

$$\mathcal{L}(Dum_{n-1}(G)) = \mathcal{L}(G) \cup \{S \cup \{n\} : S \in \mathcal{L}(G)\}$$
$$\mathcal{L}(Dum_{n-1}(G)^*) = \mathcal{L}(G^*) \cup \{S \cup \{n\} : S \in \mathcal{L}(G^*)\}$$

Proof. If we add a dummy to G, the losing sets of this game are just the losing sets of G plus all the losing sets of G with the dummy added. The other result is dual.

□

Theorem 490. $\mathcal{L}(Dom_n(G)) = \{S : S \in \mathcal{L}(G) \wedge (n \notin S)\}$ *So we take all the sets in* $\mathcal{L}(G)$ *that* n *is not in.*

$$\mathcal{L}(Dom_n(G)^*) = \mathcal{L}(Cod_n(G^*) = \{S - \{n\} : S \in \mathcal{L}(G^*) \wedge (n \in S)\}$$

Proof. I will prove this and the following theorem as a pair, after the statement of the next theorem.

□

Theorem 491. $\mathcal{L}(Cod_n(G)) = \{S - \{n\} : S \in \mathcal{L}(G) \wedge (n \in S)\}$ *So we take all the sets in* $\mathcal{L}(G)$ *that* n *is in and we knock* n *out of them.*

$$\mathcal{L}(Cod(G)^*_n) = \mathcal{L}(Dom_n(G^*)) = \{S : S \in \mathcal{L}(G^*) \wedge (n \notin S)\}$$

Proof. I prove this theorem and Theorem 490.

Let us say that $S \in \mathcal{L}(Dom_n(G))$. Then S does not win $Dom_n(G)$. So S is not a set T such that $T - \{n\}$ would have won G. So S is a set $T - \{n\}$ such that $T - \{n\}$ would have lost G. So $n \notin S$ and S loses G. This proves the first part of Theorem 490.

Let us say that $S \in \mathcal{L}(Cod_n(G))$. Then S does not win $Cod_n(G)$. So S is not a subset, T, of the set of the first $(n-1)$ voters such that $T \cup \{n\}$ would have won G. So it is a subset of the first $(n-1)$ voters such that $T \cup \{n\}$ would have lost G This proves the first part of this theorem.

The second part of each theorem is dual to the first part of the other theorem.

□

Comments 492. As we have seen, ordered pairs of simplicial complexes (with their Alexander duals) can be thought of as an order category. When they are conceived in this way we can see that the category of ordered pairs of simplicial complexes based in a space of n voters is isomorphic to C_n

Applying arrows of $\mathbf{A}$ (bloc formation, subgame and isomorphism) almost correspond to simplicial maps between the ordered pairs of complexes.

Injective maps between games do correspond to injective simplicial maps between the associated complexes. For example, given an injective $f : G \to H$, let S be a simplex in $\mathcal{L}(G)$. Assume that $f(S)$ is not a simplex in $\mathcal{L}(H)$. Then $f(S)$ must not be losing in H hence it must be winning in H. And by the conservative nature of f this means that $f^{-1}(f(S))$ must be winning for G. Since f is one-to-one then $f^{-1}(f(S)) = S$ and S is winning for G and we have a contradiction that shows the $f(S)$ must be a simplex in $\mathcal{L}(H)$.

In general we cannot be sure that the inverse image of a voter, under f, is losing. So f cannot be a simplicial map between the simplicial complexes of losing coalitions. If f is not one-to-one then we cannot assume $f^{-1}(f(S)) = S$ as we did in the last proof.

For example, let G be the game that has $\{1, 2\}$ and $\{3, 4\}$ as minimal winning coalitions. Let f be the function that is the identity on 3 and 4 and maps 1 and 2 to the same object, call it 12. $\{12\}$ is not a losing coalition of H and so f cannot be a simplicial map between $\mathcal{L}(G)$ and $\mathcal{L}(H)$; 1 and 2 have nowhere to go.

11 The Beginnings of Algebraic Game Theory

Theorem 493. *Let G be a weighted SVG, if $\mathcal{L}(G)$ (or $\mathcal{L}(G^*)$) has a non-trivial edge group then $\mathcal{L}(G)$ (or $\mathcal{L}(G^*)$) contains an empty triangle.*

Edge groups are defined in many texts on topology for example [8, P87,Theorem 4] and [10, P131,6.4].

Proof. To obtain a contradiction, let us assume that we have a weight function $w : V \to R$ where R is the set of real numbers. Of course S wins $\iff \sum_{i \in S} w(i) > q$ for some real q. (Definition 22)

Let us assume that $\mathcal{L}(G)$ has a non-trivial edge group. Let $\{(v_1, v_2), (v_2, v_3), (v_3, v_4), \ldots, (v_{n-1}, v_n), (v_n, v_1))\}$ be the smallest cycle. There must be one as the group is non-trivial. to obtain a contradiction, let us assume that $n > 3$.

It cannot be the case that (v_1, v_3) loses. If it did then $\{(v_1, v_3), (v_3, v_4), \ldots, (v_{n-1}, v_n), (v_n, v_1))\}$ would be a smaller cycle.

So (v_1, v_3) cannot be losing but (v_1, v_2) and (v_2, v_3) are. This tells us that $w(v_2) < w(v_3)$ and $w(v_2) < w(v_1)$.

On the other hand, we can see that (v_2, v_4) cannot be losing but (v_2, v_3) and (v_3, v_4) are. This tells us that $w(v_3) < w(v_4)$ and $w(v_3) < w(v_2)$.

$w(v_2) < w(v_3)$ and $w(v_3) < w(v_2)$ are clearly contradictory.

For example, the game $\{\{1, 3\}, \{2, 4\}\}$ has the losing simplicial complex with maximal sets $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$. If we assume that this is weighted then the lack of $\{1, 3\}$ and $\{2, 4\}$ in the SC tells us four things: $w(3) > w(2)$, $w(1) > w(2)$, $w(2) > w(3)$ and $w(2) > w(1)$. Clearly, these contradict in pairs. Or, to put it another way $\{1, 3\}$ and $\{2, 4\}$ win but $\{2, 3\}$ and $\{1, 4\}$ lose. This means that $w(\{1, 3\}) + w(\{2, 4\}) > w(\{2, 3\}) + w(\{1, 4\})$ or $w(\{1, 2, 3, 4\}) > w(\{1, 2, 3, 4\})$.

It is clear that G is weighted if and only if G^* is weighted.

$\square$

Theorem 494. *There is a stronger result.*

Let G be a linear SVG ([3, Definition 3.2.5]). If $\mathcal{L}(G)$ (or $\mathcal{L}(G^*)$) has a *non-trivial edge group then $\mathcal{L}(G)$ (or $\mathcal{L}(G^*)$) contains an empty triangle.*

Proof. The proof of Theorem 493 simply carries over with $v_i \leq_I v_j$ replacing $w(v_i) \leq w(v_j)$. In this case, of course, we have no way of adding the weights but the part of the proof where we added the weights was just an alternative method. To complete the proof, we need this lemma.

□

Theorem 495. *If G is a linear game then so is G^*. The desirability relation for G^* is the same as the desirability relation for G.*

Proof. Let us say that $\leq_l$ be a desirability relation on G ([3, Definition 3.2.5]). We can show that this is also a desirability relation for G^*. Assume that $a \leq_l b$. To obtain a contradiction, I will assume that $X \cup \{a\}$ wins G^* where $X \cup \{b\}$ does not win G^* (where X does not contain a and b). This means that $N - (X \cup \{a\})$ does not win G and $N - (X \cup \{b\})$ does win G. Or $(N - X - \{a, b\}) \cup \{b\}$ does not win G and $(N - X - \{a, b\}) \cup \{a\}$ does win G. This contradicts the fact that $a \leq_l b$.

□

Comments 496. In fact, we have a slightly stronger result

Theorem 497. *Let G be a linear SVG ([3, Definition 3.2.5]). If $\mathcal{L}(G)$ (or $\mathcal{L}(G^*)$) contains and empty polygon then it contains an empty triangle.*

Proof. The proof is just the same as that of Theorem 494.

□

Comments 498. There are clear parallels between the proof of Theorem 493 and trading as described in [3]. [3, Proposition 3.2.6] states that being a linear game is equivalent to being swap-robust.

We have proved a result using the fundamental group of the topological space: the first homotopy group. The homology groups can also be investigated bearing in mind the fact that the i^{th} homology group of a simplicial complex is equal to the $(|V| - i - 3)^{th}$ reduced cohomology group of the Alexander Dual of the simplicial complex ([9]).

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